MATH-F310: Differential Geometry I
- Assignment 5 -

Differential of maps and critical points

1. $\diamondsuit$ [1] Prove that a function with vanishing differential on a connected surface is constant.
2. $\diamondsuit$ Define the height function $h : (x, y, z) \mapsto z$ on the torus:

$$T := \left\{ \left(\sqrt{x^2 + y^2} - R \right)^2 + z^2 = r^2 \right\}, R > r.$$

   Find out its critical points. Explain with pictures the critical points of the function $(x, y, z) \mapsto y$.
3. $\diamondsuit$ The holomorphic map $z \mapsto z^2$ extends to a smooth map $f : S^2 \to S^2$ using stereographic projection. Compute the differential of this map at a point $p \in S^2$ and find out the critical points of f.
4. $\clubsuit$ Construct a map $f\colon T \to T$ which is a local diffeomorphism but not a diffeomorphism.
5. $\clubsuit$ For $t \in \mathbb{R}$, let $F_t : \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation of angle t along the z-axis. Prove that for all $t \in \mathbb{R}$, $f_t := F_t|_{S^2} : S^2 \to S^2$ is a diffeomorphism. Compute the differential of f_t. For any $p \in S^2$, define its flow line as $\gamma_p : \mathbb{R} \to S^2, t \mapsto f_t(p)$. Compute $\gamma_p'(t)$.
6. † Prove that the set of critical points of a smooth function on a surface is closed and hence compact if the surface is compact.
7. † Suppose S is an oriented surface and $n : S \to S^2$ is its Gauss map. Compute the differential of n.

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[1] Exercises marked by a $\diamondsuit$ will be done in class (if time permits).
Exercises marked by a $\clubsuit$ are to prepare at home for the second test.
Exercises marked by a † are extra exercises.