Assignment 2

MATH-F310: Differential Geometry I

October 6, 2022

1. **Stereographic projection**:
   (a) Let S^2 be the standard 2- dimensional sphere of radius 1 centered at the origin in $\mathbb{R}^3$. Describe the two stereographic projections from the north and south poles (call them N and S) respectively.
   (b) Say, $\phi_N : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ be one stereographic projection. And we define $\psi_N := \phi_N^{-1} : \mathbb{R}^2 \longrightarrow S^2 \setminus \{N\} \subset \mathbb{R}^3$. Show that $D\psi_N$ is injective at each point. Thus ψ_N is a parametrization at each point of $S^2 \setminus \{N\}$.
   (c) Say ψ_S is the other map: $\mathbb{R}^2 \longrightarrow S^2 \setminus \{S\}$. Prove that:
   $$\psi_{SN} := \psi_S^{-1} \circ \psi_N : \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}^2 \setminus \{0\} \text{ is given by}$$
   $$\psi_{SN}(u, v) = \frac{1}{u^2 + v^2}(u, v)$$
   (d) Any circle that does not pass through the north pole $(0, 0, 1)$, is mapped to a circle on the xy-plane: $z = 0$.
   (e) Stereographic projections preserves angles, in the sense that if two curves intersect at an angle θ on the sphere, so do their images curves on the plane $z = 0$. (*to be done after some more classes maybe*)

2. Show that the cylinder in $\mathbb{R}^3 : \{(x, y, z) : x^2 + y^2 = 1\}$ can be described locally as a graph of a function from $\mathbb{R}^2 \longrightarrow \mathbb{R}$. Hence it's a surface.

3. Consider the ellipsoid $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$. Write down local charts for this surface in $\mathbb{R}^3$. Now consider the function $(x, y, z) \mapsto x$ on this surface. Write down the coordinate representation of this function with respect to the charts.

4. Consider the hyperboloid $x^2 + \frac{y^2}{4} - \frac{z^2}{4} = 1$. For any point on the hyperboloid, find a local chart and compute the transition map between two non-equal overlapping charts. Now consider the function $(x, y, z) \mapsto y^2$ on this surface. Write down the coordinate representation of this function with respect to two different charts of your choice.

5. Let $S \subset \mathbb{R}^3$ be a surface. For any smooth function $f : S \longrightarrow \mathbb{R}$ and any $p \in S, \exists$ an open neighbourhood $W \subset \mathbb{R}^3$ around p and a smooth function $h : W \longrightarrow \mathbb{R}^3$ such that $h|_{S \cap W} = f|_W$.