Assignment 3

MATH-F310: Differential Geometry I

October 13, 2022

1. Give coordinate charts for the torus $T := S^1 \times S^1$ and prove that it's a smooth surface. We define a map

   $$f : S^1 \times S^1 \to S^1 \times S^1, (z_1, z_2) \mapsto (z_1 z_2, z_1 \overline{z_2})$$

   Compute the coordinate representations of this map and prove that it's smooth.

2. $z \mapsto z^2$ is a holomorphic and hence smooth map from $\mathbb{C} \to \mathbb{C}$. Use this map and the stereographic projections to construct a map from $S^2 \to S^2$. Compute the coordinate representations of this map and prove that it's smooth.

3. Let p be a homogeneous polynomial of degree 5 in 3-variables. Homogeneity means

   $$p(tx_1, tx_2, tx_3) = t^5 p(x_1, x_2, x_3).$$

   Prove that the set of points x, where $p(x) = a$, is a surface in $\mathbb{R}^3$, provided that $a \neq 0$.

   [Hint: Use Euler's identity for homogeneous polynomials:$\sum_{i=1}^{3} x_i \frac{\partial p}{\partial x_i} = 5 \cdot p$.]

4. Prove that a connected surface is path-wise connected.

5. Recall $\psi_N : S^2 \setminus \{N\} \to \mathbb{C}$ is the stereographic projection with respect to the north pole $\{N\}$. We define a map on S^2 in the following way:

   $$f : S^2 \setminus \{N, S\} \to S^2 \setminus \{N, S\}, x \mapsto \psi_N^{-1} \circ (z \mapsto \frac{1}{z}) \circ \psi_N$$
   $$\text{and } f(N) = S, f(S) = N.$$

   Prove that f is smooth.