Assignment 4

MATH-F310: Differential Geometry I

October 18, 2022

1. If $f : X \longrightarrow Y$ is a diffeomorphism between two surfaces, prove that at each $x \in X$, the derivative Df_x is an isomorphism of tangent spaces.

2. Let $S^2 = \{x^2 + y^2 + z^2 = 1\}$ be the unit two-sphere in $\mathbb{R}^3$. Compute the differential of the map $f : S^2 \longrightarrow \mathbb{R}; (x, y, z) \mapsto z$ at a point $(a, b, c) \in S^2$ and show that the poles are two critical points of f.

3. Define the height function $h : (x, y, z) \mapsto y$ on the torus: $\{(\sqrt{x^2 + z^2} - a)^2 + y^2 = r^2\}, a > r$. Find out its critical points. Explain with pictures the critical points of the function $(x, y, z) \mapsto z$.

4. Prove that the set of critical points of a smooth function on a surface is closed and hence compact if the surface is compact.

5. Prove that a function with vanishing differential on a connected surface is constant.

6. Describe the tangent space at a point on the sphere S^2, Compute the differential of the antipodal map: $S^2 \longrightarrow S^2, x \mapsto -x$ at a point on the sphere.