# Assignment 9

## MATH-F310: Differential Geometry I

December 8, 2022

**Preimage Theorem:** *If $n$ is a regular value of some $f \in C^\infty(M, N)$ such that $f^{-1}(n) \neq \emptyset$, then $f^{-1}(n)$ is a smooth manifold of dimension $k := \dim M - \dim N$.*

1. Assume the statement of the preimage theorem, now prove that at a point $m \in f^{-1}(n)$, $T_m(f^{-1}(n)) = \ker d_m f$.

2. Let $f : S^3 \to \mathbb{R}$ be the function $f(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2$. Show that $\frac{1}{2}$ is a regular value of $f$. Deduce that $f^{-1}(\frac{1}{2})$ is a submanifold of $S^3$ and show that this submanifold is diffeomorphic to the torus $S^1 \times S^1$.

3. This exercise is to prove that the orthogonal group $O(n) := \{A \in M(n) \,|\, AA^t = I\}$ is a smooth manifold and compute its dimension.

   a. Use exercise 1 from assignment 1 and use the fact that both the spaces:

$$M(n) : \text{the space of all } n \times n \text{ real valued matrices}$$
$$S(n) : \text{the space of all symmetric matrices inside } M(n)$$

   are vector spaces of dimension $n^2$ and $\frac{n(n+1)}{2}$ respectively. And hence the tangent space at any point of these can be canonically identified with the vector space itself.

   b. Now define a map $f : M(n) \to S(n)$ by $f(A) = AA^t$, show that $f$ is smooth and notice that $O(n) = f^{-1}(I)$. Now prove that $I$ is indeed a regular value of $f$. So, $O(n)$ is a smooth manifold of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

   c. Now prove that $T_I(O(n))$ is the set of $n \times n$ skew symmetric matrices (with or without using exercise 1).

4. a. One can define $\mathbb{RP}^2 := S^2/\sim$, where $(x, y, z) \sim (-x, -y, -z)$. Show that $\mathbb{RP}^2$ is a smooth manifold.

   b. Define a map $f : S^2 \to \mathbb{R}^4$, by

$$f(x, y, z) := (xy, xz, y^2 - z^2, 2yz)$$

   Show that it's smooth. Moreover prove that it induces a smooth map $\tilde{f} : \mathbb{RP}^2 \to \mathbb{R}^4$.

   c. Prove that the differential of $\tilde{f}$ is injective at all points of $\mathbb{RP}^2$.