Part B

In this part choose one answer from the list provided. You do not need to provide a solution or justification.
Estimated time required: 20 min.

Problem 6 (2P). The set $S := \{(x, y, z) \in \mathbb{R}^2 \mid z^4 = x^4 + y^4\}\ldots$

[ ] is a smooth surface.
[ ] is not a smooth surface, since S is disconnected.
[ ] is not a smooth surface, since S is not homeomorphic to $\mathbb{R}^2$.
[x] is not a smooth surface, since S is not locally homeomorphic to $\mathbb{R}^2$.

Problem 7 (2P). The map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = \big(x^2 - y^2,\ 2xy\big)$ is…

[ ] a diffeomorphism.
[ ] a local diffeomorphism at each point.
[x] a local diffeomorphism at each point except the origin.
[ ] none of the above.

Problem 8 (2P). Let $f\colon S_1 \to S_2$ be a smooth surjective map between two smooth surfaces. If it is only known that $d_p f$ is injective at each point $p \in S_1$, then f must be…

[ ] a diffeomorphism.
[x] a local diffeomorphism.
[ ] injective.
[ ] none of the above.

Problem 9 (2P). Chose a correct statement from the following list:…

[ ] Each smooth surface in $\mathbb{R}^3$ is orientable.
[x] Each smooth compact surface in $\mathbb{R}^3$ is orientable.
[ ] If a smooth surface $S \subset \mathbb{R}^3$ is orientable, then S is compact.
[ ] If S is non-orientable, then a unit normal field may still exist on S.

Problem 10 (2P). For an arbitrary smooth function f on a smooth surface S the following holds:

[ ] If $p \in \operatorname{supp} f$, then $f(p) \neq 0$.
[ ] If $p \notin \operatorname{supp} f$, then $f(p) \neq 0$.
[x] If $p \notin \operatorname{supp} f$, then $f(p) = 0$.
[ ] None of the above applies.

$\boxed{11\,i.}$ $\nabla H = (1,1,1)$. Hence, $(x,y,z) \in S$ is a critical pt of h if and only if $\exists \lambda \in \mathbb{R}$ s.t.

$$\nabla H = \lambda \nabla \varphi \, ,$$

where $\varphi(x,y,z) = x^2 + y^2 - z^2 - 1$. Thus,

$$\begin{cases} 2\lambda x = 1 \\ 2\lambda y = 1 \\ -2\lambda z = 1 \end{cases} \xRightarrow{\lambda \neq 0} \quad x = \frac{1}{2\lambda} \, , \; y = \frac{1}{2\lambda} \, , \; z = -\frac{1}{2\lambda} \, .$$

$$(x,y,z) \in S \implies \frac{1}{4\lambda^2} + \cancel{\frac{1}{4\lambda^2}} - \cancel{\frac{1}{4\lambda^2}} = 1 \implies$$

$\lambda = \pm \frac{1}{2}$. Hence, h has 2 critical pts, namely

$$p_1 = (1, 1, -1) \quad \text{and} \quad p_2 = (-1, -1, 1) \, .$$

$\boxed{11\,ii}$ <u>Solution 1</u>

Let $\gamma : (-\varepsilon, \varepsilon) \to S$, $\gamma(t) = (x(t), y(t), z(t))$ be a curve such that

$$\begin{matrix} \gamma(0) = p_1 \\ \dot{\gamma}(0) = v \end{matrix} \iff \begin{matrix} x(0) = 1 \, , \; y(0) = 1 \, , \; z(0) = -1 \, . \\ \dot{x}(0) = v_1 \, , \; \dot{y}(0) = v_2 \, , \; \dot{z}(0) = v_3 \, . \end{matrix}$$

$$\gamma(t) \in S \;\; \forall t \implies x(t)^2 + y(t)^2 = z(t)^2 + 1 \quad \forall t$$

$$\implies x\dot{x} + y\dot{y} - z\dot{z} = 0 \qquad (*)$$

In particular, for $t = 0$ we have

$$v_1 + v_2 + v_3 = 0 \qquad (**)$$

$$\text{Hess}_{p_1} h(v) = \frac{d^2}{dt^2}\Big|_{t=0} (h \circ \gamma(t)) = \frac{d^2}{dt^2}\Big|_{t=0} (x(t) + y(t) + z(t))$$

$$= \ddot{x}(0) + \ddot{y}(0) + \ddot{z}(0)$$

$$\frac{d}{dt}(*) : \quad x\ddot{x} + y\ddot{y} - z\ddot{z} + \dot{x}^2 + \dot{y}^2 - \dot{z}^2 = 0$$

$$\overset{t=0}{\Longrightarrow} \quad \ddot{x}(0) + \ddot{y}(0) + \ddot{z}(0) = \dot{z}(0)^2 - \dot{x}(0)^2 - \dot{y}(0)^2$$

$$= v_3^2 - v_1^2 - v_2^2$$

$$\overset{(**)}{=} (v_1 + v_2)^2 - v_1^2 - v_2^2$$

$$= 2 v_1 v_2$$

Hence, $\text{Hess}_{p_1} h$ takes both positive and negative values $\Longrightarrow$ p_1 is a saddle pt (neither loc. max. nor loc. minimum).
The other crit. pt can be handled in a similar manner.

<u>Solution 2</u>

Consider the following parametrization of S:

$$\Psi(x, y) := (x, y, -\sqrt{x^2 + y^2 - 1}),$$

where $(x, y) \in V = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 > 1 \}$.
Notice that $\Psi(1,1) = p_1$.

We have

$$h \circ \varphi (x,y) = H \circ \varphi (x,y) = x + y - \sqrt{x^2+y^2-1}$$

$$\frac{\partial^2}{\partial x^2} h \circ \varphi = -\frac{\partial}{\partial x} \frac{x}{\sqrt{x^2+y^2-1}} = - \frac{\sqrt{x^2+y^2-1} - x \frac{x}{\sqrt{x^2+y^2-1}}}{x^2+y^2-1}$$

$$= \frac{x^2+y^2-1-x^2}{(x^2+y^2-1)^{3/2}} = \frac{y^2-1}{(x^2+y^2-1)^{3/2}}$$

$$\frac{\partial^2}{\partial x \partial y} h \circ \varphi = -\frac{\partial}{\partial y} \frac{x}{\sqrt{x^2+y^2-1}} = \frac{1}{2} \frac{2y}{(x^2+y^2-1)^{3/2}} = \frac{y}{(x^2+y^2-1)^{3/2}}$$

$$\frac{\partial^2}{\partial y^2} h \circ \varphi = \frac{x^2-1}{(x^2+y^2-1)^{3/2}}$$

Hence, $\mathrm{Hess}_{(1,1)} (h \circ \varphi) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow$ $(1,1)$ is a saddle pt of $h \circ \varphi$

$\Rightarrow$ $(1,1,-1)$ is a saddle pt of h.

One can determine the type of p_2 by considering the parametrization

$$\psi(x,y) = (x, y, \sqrt{x^2+y^2-1}).$$

$\boxed{12}$ $S = f^{-1}(c) = g^{-1}(d)$

$\forall p \in S \quad \nabla f(p), \nabla g(p) \in T_pS^{\perp}$, $\dim T_pS^{\perp} = 1$.

$\nabla g(p) \neq 0 \implies \exists \lambda = \lambda(p) \in \mathbb{R}$ s.t. $\nabla f(p) = \lambda(p) \nabla g(p)$

Pick $p \in S$. Since $\nabla g(p) \neq 0$, without loss of generality we can assume $\frac{\partial g}{\partial x}(p) \neq 0$.

$\frac{\partial f}{\partial x}(q) = \lambda(q) \frac{\partial g}{\partial x}(q) \quad \forall q \in S$

a nbhd of p s.t. $\frac{\partial g}{\partial x}(q) \neq 0$ $\forall q \in U$.

$\lambda(q) = \dfrac{\partial f / \partial x \,(q)}{\partial g / \partial x \,(q)}$, $q \in U$.

Both $\frac{\partial f}{\partial x}\big|_S$ and $\frac{\partial g}{\partial x}\big|_S$ are smooth f-ns on S as restrictions of smooth f-ns.

Moreover, $\frac{\partial g}{\partial x}$ does not vanish on U, hence λ is smooth on U. Since smoothness is a local property, λ is smooth everywhere on S.

$\boxed{13}$ See P. 20 of Part 4 of the lecture notes (Step 1 + Step 2).

$\boxed{14}$ Let $\varphi : V \to U \subset S$ be a parametrization of S s.t. $p \in U$. Assume $\varphi(0) = p$.

Then 0 is a critical pt of $F := f \circ \varphi$.

Moreover, for any smooth curve $\beta : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ s.t. $\beta(0) = 0$ we have

$$\frac{d^2}{dt^2}\Big|_{t=0} F \circ \beta(t) = \mathrm{Hess}_0 F(\dot\beta(0))$$

Denoting $\gamma = \varphi \circ \beta$, we obtain

$$\frac{d^2}{dt^2}\Big|_{t=0} \left(F \circ \beta(t)\right) = \frac{d^2}{dt^2}\Big|_{t=0} \left(f \circ \gamma(t)\right) = \mathrm{Hess}_p f\,(\dot{\gamma}(0))$$

Furthermore, $\dot{\gamma}(0) = D_0 \varphi(\dot{\beta}(0)) \neq 0$ provided $\dot{\beta}(0) \neq 0$ since $D_0\varphi$ is injective. Hence,

$\mathrm{Hess}_p f$ is positive-def. $\Rightarrow$ $\mathrm{Hess}_0 F$ is positive-definite

$\Rightarrow$ 0 is a pt of loc. minimum for F

$\Rightarrow$ p is a pt of loc. min. for f.

$\boxed{15}$ Since S is compact, $\exists R > 0$ s.t.

$S \subset B_R(0)$ ← the open ball of radius R centered at the origin.

Pick a non-zero vector w in $\mathbb{R}^3$, for example $w = (1,0,0)$. By the compactness of S, $\exists T > 0$ such that

- $S \subset B_R(tw) \quad \forall t \in [0, T)$
- $S \cap \partial \overline{B}_R(\underbrace{Tw}_{=:a}) \neq \emptyset$.

Pick any $p \in S \cap \partial\overline{B}_R(a)$.

Let $\gamma : (-\varepsilon, \varepsilon) \to S$ be any curve s.t. $\gamma(0) = p$.

By construction, $\gamma(0) = p$ and $\gamma(t) \in \overline{B}_R(a)$, that is the function $t \mapsto \|\gamma(t) - a\|^2$ has a loc. maximum at $t = 0$. Hence,

$$\frac{d}{dt}\Big|_{t=0} \langle \gamma(t)-a, \gamma(t)-a \rangle$$

$$= \langle \dot\gamma(0), p-a \rangle + \langle p-a, \dot\gamma(0) \rangle = 2\langle \dot\gamma(0), p-a \rangle = 0$$

$$\Rightarrow \quad T_p S \subset T_p S^2_R(a)$$
$$\qquad\qquad\qquad\quad \| $$
$$\qquad\qquad\qquad \partial \bar B_R(a)$$

Since $\dim T_p S = 2 = \dim T_p S^2_R(a)$, we have

$$T_p S = T_p S^2_R(a) = (p-a)^\perp$$

Moreover, if $\dot\gamma(0) = v$, then

$$\frac{d^2}{dt^2}\Big|_{t=0} \langle \gamma(t)-a, \gamma(t)-a \rangle = 2\langle \ddot\gamma(0), p-a \rangle + 2\langle \dot\gamma(0), \dot\gamma(0) \rangle$$

$$= 2\langle \ddot\gamma(0), p-a \rangle + 2|v|^2 \leq 0 \qquad (*)$$

Denote $n := \frac{p-a}{|p-a|}$, which is a unit normal vector both to S and $S^2_R(a)$ at p.
Let h_n be the height f-n on S in the direction of n. Then

$$\underset{\displaystyle -\mathrm{II}^S(v)}{\underset{\|}{\mathrm{Hess}_p h_n(v)}} = \frac{d^2}{dt^2}\Big|_{t=0} \langle \gamma(t), n \rangle = \langle \ddot\gamma(0), n \rangle \overset{(*)}{\leq} -\frac{1}{|p-a|}|v|^2$$

$\Rightarrow$ Each eigenvalue of the Gauss map at p is $\geq \frac{1}{|p-a|}$ $\Rightarrow$ $K(p) \geq \frac{1}{|p-a|^2} > 0$.