Université libre de Bruxelles
Autumn 2021

Mock Exam

in Differential Geometry I

13th November 2021

Last name:

First name:

Please notice the following:

- ◇ You may bring only a pen and a bottle of water (a drink) to the exam.
- ◇ Please do not bring paper to the exam.
- ◇ **Please begin only after an invitation!**

Please fill in only above this line!

1–10	11–20	21	22	23	24	25	26	$\sum$

Mark:

Part A

Please write your answers in this part directly below each problem.

Problem 1 (1P). Formulate the definition of a topological manifold.

Problem 2 (1P). Formulate the definition of a smooth atlas on a topological manifold.

Problem 3 (1P). Formulate the definition of a smooth map between two smooth manifolds..

Problem 4 (1P). Complete the following definition:

A smooth map $f\colon M \to N$ is said to be a diffeomorphism, if the following holds:

Problem 5 (1P). Complete the following definition:

A point $m \in M$ is said to be a critical point of a map $f\colon M \to N$, if ...

Problem 6 (1P). Complete the following definition:

A point $n \in N$ is said to be a regular value of a map $f\colon M \to N$, if ...

Problem 7 (1P). Complete the following definition:

The tangent space of a manifold M at a point m consists of classes $[\gamma]$ of smooth curves in M through m, where

$\gamma_1 \sim \gamma_2 \quad \Longleftrightarrow$

Problem 8 (1P). Define the notion of a derivation on a smooth manifold M at a point m.

Problem 9 (1P). Formulate the theorem about preimages of regular values.

Problem 10 (1P). Formulate Whitney's embedding theorem.

Part B

In this part choose one answer from the list provided. You do not need to provide a solution or justification.

Problem 11 (1P). The set $M := \{(x, y) \in \mathbb{R}^2 \mid y = 0 \text{ or } y = x^2\}\dots$

[ ] is a smooth manifold.
[ ] is not a smooth manifold, since M is non-Hausdorff.
[ ] is not a smooth manifold, since M is not second countable.
[ ] is not a smooth manifold, since M is not locally homeomorphic to an Euclidean space.

Problem 12 (1P). The map $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^5$ is…

[ ] a diffeomorphism.
[ ] a homeomorphism, which is not a diffeomorphism.
[ ] continuous, however f^{-1} is discontinuous at the origin.
[ ] non-bijective, that is f^{-1} does not exist.

Problem 13 (1P). The map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = \big(x^2 - y^2,\ 2xy\big)$ is…

[ ] a diffeomorphism.
[ ] a local diffeomorphism at each point.
[ ] a local diffeomorphism at each point except the origin.
[ ] none of the above.

Problem 14 (1P). For a fixed point $m \in M$, the map $d\colon C^\infty(M) \to \mathbb{R}$, $d(f) = f(m)$ is…

[ ] a derivation.
[ ] is not a derivation, since d is non-linear over $\mathbb{R}$.
[ ] is not a derivation, since there exist some $f, g \in C^\infty$ such that $d(fg) \neq d(f) \cdot d(g)$.
[ ] not a derivation for a reason not listed above.

Problem 15 (1P). The subset $M := \left\{ \left(x, \sin \frac{1}{x} \right) \mid x > 0 \right\} \subset \mathbb{R}^2$ is…

[ ] a submanifold.
[ ] not a submanifold, since M is non-compact.
[ ] not a submanifold, since M does not admit adapted charts.
[ ] not a submanifold for a reason not listed above.

The real exam will contain ∼5 more multiple choice problems in this part totaling to approximately 10 problems.

Part C

Please write your solutions to the problems in Parts C and D on the blank sheets provided.

Problem 21 (10P). Construct a smooth atlas on $\mathbb{RP}^2$.

Problem 22 (10P). Define $F\colon \mathbb{R}^3 \to \mathbb{R}$ by $F(x, y, z) = x + y + z$. Determine all critical points of $f\colon M \to \mathbb{R}$, where

$$M := \left\{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2 + 1 \right\} \qquad \text{and} \qquad f = F\big|_M .$$

Problem 23 (10P). Let (U, φ) be a chart on a smooth manifold M, where $\varphi = (x_1, \ldots, x_k)$. Show that for each $m \in U$ the k-tuple

$$\big(\partial_1(m), \ldots, \partial_k(m) \big) \qquad \text{with} \qquad \partial_j(m) f = \frac{\partial}{\partial x_j}\Big|_{x=\varphi(m)} \big(f \circ \varphi^{-1}(x) \big)$$

is a basis of $Der_m M \cong T_m M$.

Part D

Problem 24 (10P). Let $\mathcal{U} = \{ (U_\alpha, \varphi_a) \mid \alpha \in A \}$ be an atlas on a topological manifold M. Show that $V \subset M$ is open if an only if $\varphi_\alpha(V \cap U_\alpha) \subset \mathbb{R}^k$ is open.

Problem 25 (3+3+4 P). Show that each of the following maps is a diffeomorphism:

(i) $f\colon S^3 \to \mathrm{SU}(2), \qquad f(z_1, z_2) = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}.$

(ii) $g\colon \mathrm{O}(n) \to \mathrm{O}(n), \qquad g(A) = A^{-1}.$

(iii) $h\colon \mathrm{SL}(n) \to \mathrm{SL}(n), \qquad h(A) = A^{-1}.$

Problem 26 (10P). Show that the image of the map

$$f\colon S^2 \to \mathbb{R}^6, \qquad f(x,y,z) = \big(\, x^2,\ y^2,\ z^2,\ \sqrt{2}\,yz,\ \sqrt{2}\,zx,\ \sqrt{2}\,xy\,\big)$$

is a submanifold diffeomorphic to $\mathbb{RP}^2$.