Université libre de Bruxelles
Autumn 2022

Mock Exam

in Differential Geometry I

28th November 2022

Last name:

First name:

Please notice the following:

- ◇ You may bring only a pen and a bottle of water (a drink) to the exam.
- ◇ Please do not bring any paper to the exam.
- ◇ This mock exam covers only the material about smooth surfaces. The real exam will contain also questions and/or problems about smooth manifolds.
- ◇ To test yourself, try to solve (and write down clearly!) as much as possible in 3 hrs.

Please fill in only above this line!

HW	1–5	5–10	11	12	13	14	15	$\sum$

Mark:

Part A

Please write your answers in this part directly below each problem.
Estimated time required: 20 min.

Problem 1 (2P). Formulate the definition of a smooth surface.

Problem 2 (2P). Formulate the proposition about the representability of a smooth surface as a zero locus of a function.

Problem 3 (2P). Complete the following definition: Let S_1 and S_2 be two smooth surfaces. A map $f\colon S_1 \to S_2$ is said to be smooth, if ...

Problem 4 (2P). Let f be a smooth map between smooth surfaces S_1 and S_2. Define the differential of f at a point $p \in S_1$.

Problem 5 (2P). Let $h\colon S_1 \to S_2$ be a map between smooth compact surfaces. Formulate the theorem relating integrals of a smooth function $f\colon S_2 \to \mathbb{R}$ and $f \circ h$.

Part B

In this part choose one answer from the list provided. You do not need to provide a solution or justification.

Estimated time required: 20 min.

Problem 6 (2P). The set $S := \{(x, y, z) \in \mathbb{R}^2 \mid z^4 = x^4 + y^4\}\dots$

[ ] is a smooth surface.

[ ] is not a smooth surface, since S is disconnected.

[ ] is not a smooth surface, since S is not homeomorphic to $\mathbb{R}^2$.

[ ] is not a smooth surface, since S is not locally homeomorphic to $\mathbb{R}^2$.

Problem 7 (2P). The map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = \big(x^2 - y^2,\ 2xy\big)$ is...

[ ] a diffeomorphism.
[ ] a local diffeomorphism at each point.
[ ] a local diffeomorphism at each point except the origin.
[ ] none of the above.

Problem 8 (2P). Let $f\colon S_1 \to S_2$ be a smooth surjective map between two smooth surfaces. If it is only known that $d_p f$ is injective at each point $p \in S_1$, then f must be...

[ ] a diffeomorphism.
[ ] a local diffeomorphism.
[ ] injective.
[ ] none of the above.

Problem 9 (2P). Chose a correct statement from the following list:...

[ ] Each smooth surface in $\mathbb{R}^3$ is orientable.
[ ] Each smooth compact surface in $\mathbb{R}^3$ is orientable.
[ ] If a smooth surface $S \subset \mathbb{R}^3$ is orientable, then S is compact.
[ ] If S is non-orientable, then a unit normal field may still exist on S.

Problem 10 (2P). For an arbitrary smooth function f on a smooth surface S the following holds:

[ ] If $p \in \operatorname{supp} f$, then $f(p) \neq 0$.
[ ] If $p \notin \operatorname{supp} f$, then $f(p) \neq 0$.
[ ] If $p \notin \operatorname{supp} f$, then $f(p) = 0$.
[ ] None of the above applies.

Part C

Please write your solutions to the problems in Parts C and D on the blank sheets provided.
Estimated time required: 1 hr 20 min.

Problem 11 (5+5P). Define $H\colon \mathbb{R}^3 \to \mathbb{R}$ by $H(x,y,z) = x + y + z$.

(i) Determine all critical points of $h\colon S \to \mathbb{R}$, where

$$S := \big\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2 + 1\big\} \qquad \text{and} \qquad h = H\big|_S.$$

(ii) Determine which critical points of h are points of local maxima/minima.

Problem 12 (10P). Show that if a surface $S \subset \mathbb{R}^3$ is represented both as $f^{-1}(c)$ and $g^{-1}(d)$ for some $f, g \in C^\infty(\mathbb{R}^3)$ such that $\nabla f(p) \neq 0$ and $\nabla g(p) \neq 0$ for any $p \in S$, then there exists a smooth nowhere vanishing function $\lambda \in C^\infty(S)$ such that $\nabla f(p) = \lambda(p) \nabla g(p)$ holds for all $p \in S$.

Problem 13 (10P). Let S be a smooth surface in $\mathbb{R}^3$ with positive Gauss curvature. Show that the Gauss map of S is surjective.

Part D

Estimated time required: 1 hr.

Problem 14 (10P). Let $S \subset \mathbb{R}^3$ be a smooth surface. Let p be a critical point of a smooth function $f\colon S \to \mathbb{R}$. Show that if the Hessian of f at p is positive-definite, then f has a local minimum at p.

Problem 15 (10P). Show that there are no smooth compact surfaces, whose Gauss curvature is everywhere non-positive.