List of Problems in Global Analysis

1. Let M be a closed oriented Riemannian manifold. Show that any solution $\omega \in \Omega^k(M)$ of the equation $\Delta\omega = d\eta$, where $\eta \in \Omega^{k-1}(M)$, is closed.

2. Prove that any cohomology class in $H^1_{dR}\big(\mathbb{R}^2 \setminus \{0\}\big)$ is represented by a harmonic 1-form.

3. Prove that on any closed connected oriented Riemannian n-manifold, any harmonic n-form is proportional to the volume form.

4. Let Σ be a Riemann surface.

   (i) Show that for any holomorphic $(1,0)$ form ζ, the real 1-forms $\operatorname{Re}\zeta$ and $\operatorname{Im}\zeta$ are harmonic.

   (ii) Show that for any real harmonic 1-form ω there exists a holomorphic $(1,0)$ form ζ such that $\operatorname{Re}\zeta = \omega$.

5. Prove that the wedge-product of harmonic forms does not need to be harmonic (*Hint:* Take a compact Riemann surface Σ of genus ≥ 2. Pick a non-trivial holomorphic $(1,0)$ form ζ. Show that $\operatorname{Re}\zeta \wedge \operatorname{Im}\zeta \neq 0$ must vanish somewhere and therefore cannot be harmonic.)

6. Prove that the tangent bundle of the 2-sphere is non-trivial.

7. Denote
$$L = \big\{([z], w) \in \mathbb{CP}^1 \times \mathbb{C}^2 \mid w = 0 \text{ or } [w] = [z]\big\}.$$
Define the projection map $\pi\colon L \to \mathbb{CP}^1$ by $\big([z], w\big) \mapsto [z]$. Show that L is a complex vector bundle of rank 1 over $\mathbb{CP}^1 \cong S^2$. This is called the tautological line bundle of $\mathbb{CP}^1$.

8. Let L be a complex line bundle bundle, that is a complex vector bundle of rank 1, over S^2 such that L admits a trivialization σ_N over $S^2 \setminus \{N\}$ and a trivialization σ_S over $S^2 \setminus \{S\}$, where $N = -S$ is the northern pole[1]. This yields a map $g\colon S^2 \setminus \{S, N\} \to \mathbb{C}^*$ defined by
$$\sigma_S(m) = g(m)\sigma_N(m).$$
The degree of the map $g/|g|\colon S^1 \to S^1$, where the source $S^1 \subset S^2 \setminus \{S, N\}$ is thought of as the equator, is called the degree of L. Show that the following holds:

   (i) The degree of a complex line bundle is well-defined and depends on the isomorphism class of L only.

   (ii) The degree of the tautological bundle equals -1.

   (iii) The degree of T^*S^2 equals 2. Here T^*S^2 is viewed as a complex line bundle as follows: The Hodge operator on T^*S^2 satisfies $*^2 = -id$. Hence, elements of T^*S^2 can be multiplied by complex numbers: $(a + bi)\cdot\omega := a\omega + b * \omega$.

   (iv) $\deg(L_1 \otimes L_2) = \deg L_1 + \deg L_2$.

   (v) $\deg L^* = -\deg L$, where $L^* = \operatorname{Hom}(L, \underline{\mathbb{C}})$ is the dual line bundle.

   (vi) For any integer n there exists a complex line bundle L_n such that $\deg L_n = n$.

[1] One can show that in fact any vector bundle has this property.

(vii) Two line bundles are isomorphic if and only if their degrees are equal.

(viii) Prove that the tangent bundle of S^2 is non-trivial.

9. Show that any function $f \in H^1(0,1)$ is continuous without using the Sobolev embedding theorem.

10. Show that the function

    (i) $f(x) = |x|$ belongs to $H^1(-1,1)$;

    (ii) $f(x) = |x|^{1/2}$ does not belong to $H^1(-1,1)$.

11. For which values of $a \in \mathbb{R}$ does the function $f(x) = |x|^a$ belong to $H^k(\mathbb{R}^n)$?

12. Show that there exists a function $f \in H^1(\mathbb{R}^2)$, which is not continuous.

13. Show that the operator

$$L\colon C^\infty(\mathbb{R}^3;\mathbb{H}) \to C^\infty(\mathbb{R}^3;\mathbb{H}), \qquad Lu = i\,\partial_x u + j\,\partial_y u + k\,\partial_z u$$

    is elliptic, where $\mathbb{H}$ denotes the algebra of quaternions.

14. Is the bi-Laplacian $u \mapsto \Delta(\Delta u)$, $u \in C^\infty(\mathbb{R}^n)$, an elliptic operator? Is $d+d^*\colon \Omega^k(M) \to \Omega^{k+1}(M)\oplus\Omega^{k-1}(M)$ elliptic? Is $d+d^*\colon \Omega^{\mathrm{even}}(M) \to \Omega^{\mathrm{odd}}(M)$ elliptic, where $\Omega^{\mathrm{even}}(M) := \Omega^0 \oplus \Omega^2 \oplus \dots$?

15. Show that any pseudo-differential operator acting on $C_0^\infty(\mathbb{R}^n)$, say, is an integral operator, that is of the form

$$u \mapsto \int_{\mathbb{R}^n} K(x,y)u(y)\,dy.$$

    Compute K for the inverse of the standard Laplacian on $\mathbb{R}^n$.

16. Let

$$\Gamma(E_0) \xrightarrow{L_0} \Gamma(E_1) \xrightarrow{L_1} \Gamma(E_2) \qquad (1)$$

    be a complex, where both L_0 and L_1 are differential operators. Show that (1) is an elliptic complex if and only if the operator $L_1 + L_0^*\colon \Gamma(E_1) \to \Gamma(E_2) \oplus \Gamma(E_0)$ is elliptic.

17. Prove that a bounded linear operator $T\colon H_1 \to H_2$, where H_1 and H_2 are Hilbert spaces, is Fredholm if and only if there exist bounded linear maps $S_1, S_2\colon H_2 \to H_1$ such that

$$S_1 \circ T = \mathrm{id}_{H_1} + R_1 \qquad \text{and} \qquad T \circ S_2 = \mathrm{id}_{H_2} + R_2,$$

    where both R_1 and R_2 are compact.