Riemann Surfaces

Winter semester 2017/2018
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1st exercise sheet

Exercise 1 (2P)**.** Decide whether Σ is a Riemann Surface, where

- $\Sigma = \{(x, y) \in \mathbb{C}^2 \mid y^3 + 3y + 1 = x\}$,
- $\Sigma = \{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 = 1\}$,
- $\Sigma = \{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 = 0\}$,
- $\Sigma = \{(x, y) \in \mathbb{C}^2 \mid y^2 = x(x - 1)^2\}$.

Exercise 2 (2P)**.** Show that

1. $\mathbb{H}$ (the upper half-plane) and D (the unit disc in $\mathbb{C}$) are biholomorphic,
2. $\mathbb{H}$ and $\mathbb{C}$ are not biholomorphic.

Exercise 3 (2P)**.** Define an equivalence relation on $\mathbb{C}^{n+1} \setminus \{0\}$ as follows:
$(z_0, \dots, z_n) \sim (z'_0, \dots, z'_n)$ iff there is $\lambda \in \mathbb{C} \setminus \{0\}$ such that $z'_j = \lambda z_j$ for all $j \in \{0, \dots, n\}$.

1. Show that the *projective space* $\mathbb{C}P^n := \mathbb{C}^{n+1} \setminus \{0\} / \sim$ is a complex manifold of dimension n.
2. Decide for which $\lambda \in \mathbb{C}$ the set $\Sigma_n = \{[x : y : z] \in \mathbb{C}P^2 \mid y^2 z = x(x - z)(x - \lambda z)\}$ is a Riemann surface.

Exercise 4 (4P)**.** Prove the implicit function theorem for holomorphic functions:
Let $F\colon \mathbb{C}^2 \to \mathbb{C}$ be a holomorphic function s.t. at some point $(z_0, w_0) \in \mathbb{C}$ we have $F(z_0, w_0) = 0$ and $\frac{\partial F}{\partial w}(z_0, w_0) \neq 0$. Then there is a disc $D_1 \subset \mathbb{C}$ centered at z_0 and a disc $D_2 \subset \mathbb{C}$ centered at w_0 such that

$$\{(z, w) \mid F(z, w) = 0\} \cap (D_1 \times D_2) = \{(z, \phi(z)) \mid z \in D_1\},$$

where $\phi\colon D_1 \to D_2$ is a holomorphic function with $\phi(z_0) = w_0$.

Return: Friday, 2018/11/02, at 10 am before the lecture.