Riemann Surfaces

Winter semester 2017/2018
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2nd exercise sheet

Exercise 1 (1P)**.** Show that

$$E := \{[x : y : z] \in \mathbb{C}P^2 \mid y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3\}$$

is connected without using the fact that E can be identified with a complex torus.

Exercise 2 (2P)**.**

1. For $p_k \colon \mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z^k$, where $k \in \mathbb{Z} \setminus \{0\}$, compute the induced map $(p_k)_* \colon \pi_1(\mathbb{C}^*) \to \pi_1(\mathbb{C}^*)$.
2. Let $p \colon X \to Y$ be a covering, where both X and Y are path-connected. Prove the following: If $p_* \colon \pi_1(X) \to \pi_1(Y)$ is surjective, then p is a homeomorphism.

Exercise 3 (2P)**.** Let $g_2, g_3 \in \mathbb{R}$ be such that the polynomial $q(x) = 4x^3 - g_2 x - g_3$ has three distinct roots. Let $Q(x)$ be a primitive of $q(x)^{-1/2}$. Show that the inverse function Q^{-1} can be computed in terms of a Weierstraß $\wp$-function.
(Assume the following fact: For any $g_2, g_3 \in \mathbb{C}$ s.t. $q(x)$ has three distinct roots there is a lattice $\Lambda \subset \mathbb{C}$ s.t. $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$).

Exercise 4 (5P)**.** Let Λ and $\tilde{\Lambda}$ be two lattices in $\mathbb{C}$.

1. Show that any holomorphic map $F \colon \mathbb{C} \to \mathbb{C}/\tilde{\Lambda}$, which is doubly periodic with respect to Λ, is of the form
$$F(z) = [az + b]$$
for some $a, b \in \mathbb{C}$.
2. Let $\tilde{\psi} \colon \mathbb{C}/\tilde{\Lambda} \to \mathbb{C}$ be a local inverse of the projection $\tilde{\pi} \colon \mathbb{C} \to \mathbb{C}/\tilde{\Lambda}$ (in particular, $\tilde{\psi}$ is defined on an open subset of $\mathbb{C}/\tilde{\Lambda}$ only). Show that the derivative of $\tilde{\psi} \circ F$ gives rise to a well-defined holomorphic map $\mathbb{C} \to \mathbb{C}$.
3. Prove that a biholomorphic map $F \colon \mathbb{C}/\Lambda \to \mathbb{C}/\tilde{\Lambda}$ must be of the form $[z] \mapsto [az + b]$ for some $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$. Prove also that $a\Lambda \subset \tilde{\Lambda}$.
4. Provide an example of two complex tori which are not biholomorphic.

Return: Friday, 2018/11/16, at 2 pm before the lecture.