Riemann Surfaces

Winter semester 2017/2018
Dr. A. Haydys

3rd exercise sheet

Exercise 1 (1P)**.** Denote $H := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ and $D^* := \{z \in \mathbb{C} \mid 0 < |z| < 1\}$. Show that

$$\exp\colon H \to D^*,\ \exp(z) = e^z$$

is the universal covering of D^*.

Exercise 2 (2P)**.**

(i) Show that any holomorphic differential η on a Riemann surface is closed, i.e., $d\eta = 0$.

(ii) Show that any holomorphic differential on S^2 vanishes.

Hint: Take a holomorphic differential on $\mathbb{C}$ and analyse its behavior under coordinate change.

Exercise 3 (3P)**.** Define $\omega \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$ as follows: $\omega = \frac{x\,dy - y\,dx}{x^2+y^2}$. Show that the following holds:

1. ω is closed.
2. ω is not exact, i.e. there is no smooth function $f\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ such that $\omega = df$.
3. For any closed $\eta \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$ there is a unique number $a \in \mathbb{R}$ such that $\eta - a\omega$ is exact.

Exercise 4 (4P)**.** Let $z = x + iy$ be a local holomorphic coordinate on a Riemann Surface Σ. A real-valued function f on Σ is called harmonic, if $\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

1. Show that the notion of harmonicity is coordinate-independent.
2. Let ω be a real 1-form on Σ. In coordinates (x, y) write $\omega = a\,dx + b\,dy$ and define $*\omega := -b\,dx + a\,dy$. Show that $*\omega$ does not depend on the choice of the holomorphic coordinate z.
3. We say that ω is harmonic, if *locally* $\omega = df$ for some harmonic function f (f does not need to exist globally on Σ). Show that ω is harmonic if and only if $d\omega = 0$ and $d(*\omega) = 0$. Check that the form ω defined in Exercise 3 is harmonic (but not exact).

Return: Friday, 2018/11/30, at 2 pm before the lecture.