Riemann Surfaces

Winter semester 2017/2018
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4th exercise sheet

Exercise 1 (1P)**.** Compute the degree of the antipodal map

$$A\colon S^n \to S^n,\ A(p) = -p.$$

Hint: Consider an n-form ω s.t. $\int_{S^n} \omega = 1$ and $\operatorname{supp}\omega$ is contained in a small neighborhood of a point.

Exercise 2 (2P)**.** Denote $\mathbb{R}P^2 := S^2/\{\pm 1\}$, where $(-1)\cdot p = -p$ (the antipodal point). Show that $b_1(\mathbb{R}P^2) = 0$.

Exercise 3 (3P)**.**

- Let $f\colon \Sigma_1 \to \Sigma_2$ and $f\colon \Sigma_2 \to \Sigma_3$ be two non-constant holomorphic maps between compact Riemann surfaces. Show that

$$\deg(g \circ f) = \deg g \cdot \deg f. \tag{1}$$

- Prove (1) for any smooth maps f and g using the following weak form of Sard's theorem: For any smooth map $f\colon M \to N$ the set of regular values is open and dense in N.

Exercise 4 (4P)**.** Let $P(z,w)$ be a polynomial in two variables ob bidegree (d_1, d_2), i.e. P is a polynomial of degree d_1 in the z-variable and degree d_2 in the w-variable. Assume $\Sigma = \{P(z,w) = 0\} \subset \mathbb{C}\times\mathbb{C}$ is a smooth Riemann surface. Moreover, assume there is a smooth Riemann surface

$$\begin{aligned}\overline{\Sigma} \subset \mathbb{C}P^1 \times \mathbb{C}P^1 &= (\mathbb{C}P^1 \setminus \{\infty\} \cup \{\infty\}) \times (\mathbb{C}P^1 \setminus \{\infty\} \cup \{\infty\}) \\ &= (\mathbb{C}\times\mathbb{C}) \cup (\{\infty\}\times\mathbb{C}) \cup (\{\mathbb{C}\times\infty\}) \cup \{\mathrm{pt}\},\end{aligned}$$

s.t. $\overline{\Sigma} \cap (\mathbb{C}\times\mathbb{C}) = \Sigma$.
Prove the equality $g_{\overline{\Sigma}} = (d_1 - 1)(d_2 - 1)$.
Hint: construct a suitable meromorphic differential on $\overline{\Sigma}$.

Return: Friday, 2018/12/14, at 2 pm.