Riemann Surfaces

Winter semester 2018/2019
Dr. A. Haydys

5th exercise sheet

Exercise 1 (3P)**.**

(i) Prove that any compact Riemann surface of genus zero is biholomorphic to $\mathbb{CP}^1$.

(ii) Prove that any compact Riemann surface of genus one is biholomorphic to a complex torus $\mathbb{C}/\Lambda$.

Exercise 2 (7P)**.** Let η be a meromorphic differential on a Riemann surface Σ.

(i) Let $p \in \Sigma$ be a pole of η. Choose a local holomorphic coordinate z centered at p and expand η into a Laurant series:

$$\eta = \Big(\sum_{k=-N}^{\infty} a_k z^k \Big)\, dz.$$

Show that $\operatorname{res}_p \eta := a_{-1}$ does not depend on the choice of z.

(ii) Let D be a (small) disc containing p as an interior point. Show that $\operatorname{res}_p \eta = \frac{1}{2\pi i} \int_{\partial D} \eta$.

(iii) Prove the following: If Σ is compact and η is holomorphic outside of $\{p_1, \ldots, p_n\}$, then $\sum_{k=1}^{n} \operatorname{res}_{p_k} \eta = 0$.

(iv) With the help of the Main Theorem prove the following statement: Let $p_1, \ldots, p_n$ be $n \geq 2$ distinct points on a compact Riemann surface Σ. Let $c_1, \ldots, c_n$ be any complex numbers such that $\sum c_k = 0$. Then there is a meromorphic differential η on Σ such that η is holomorphic on $\Sigma \setminus \{p_1, \ldots, p_n\}$, η has at most a simple pole at each p_k, and $\operatorname{res}_{p_k} \eta = c_k$ for each $k \leq n$.

(v) Show that (iv) implies existence of meromorphic functions on compact Riemann surfaces.

Return: Friday, 2019/01/18, at 2 pm in room 130.