Riemann Surfaces

Winter semester 2018/2019
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6th exercise sheet

Exercise 1 (2P). Let

$$\Sigma := \{ \, y^2 = (x-\lambda_1)\cdot \ldots \cdot (x-\lambda_n) \, \} \subset \mathbb{CP}^2,$$

where all λ_j are distinct. Show that each $f_j := (x-\lambda_j)^{-1}|_\Sigma$ is a meromorphic function on Σ with a unique pole of order 2 at $(\lambda_j, 0)$. In particular, Σ is hyperelliptic. What is the genus of Σ? If $n = 3$ (so that Σ is an elliptic curve), can you represent f_j via known functions?

Exercise 2 (3P). Let $D = \{p_1, \ldots, p_n\}$ be a finite subset of a Riemann surface Σ. Show that ζ_D, which is the obstruction to the existence of meromorphic functions with at most simple poles along D, is well-defined as a map

$$\bigoplus_{p \in D} T_p\Sigma \longrightarrow H^{0,1}(\Sigma).$$

Exercise 3 (5P). For a point $p \in \Sigma$ and a positive integer n denote

$$H^0(np) := \big\{ f \colon \Sigma \setminus \{p\} \to \mathbb{C} \text{ holom. } \mid \operatorname{ord}_p f \geq -n \big\}, \qquad h^0(np) := \dim H^0(np),$$
$$H^0\big(K - np\big) := \big\{ \omega \in H^{0,1}(\Sigma) \mid \operatorname{ord}_p \omega \geq n \big\}, \qquad h^0(K-np) := \dim H^0\big(K - np\big).$$

In other words, $H^0(np)$ consists of meromorphic functions on Σ with a pole only at p of order at most n, whereas $H^0\big(K-np\big)$ consists of holomorphic differentials having a zero at p of multiplicity at least n.
The Riemann–Roch formula in this case reads:

$$h^0(np) - h^0(K - np) = n - g + 1.$$

Let $\Sigma = E$ be an elliptic curve.

(i) With the help of the Riemann–Roch formula, show that for any given $p \in E$ there is a meromorphic function f on E with a double pole at p.

(ii) Show that $\operatorname{Res}_p f = 0$.

(iii) Show that $af + b$ coincides with the Weierstraß $\wp$-function for suitable $a, b \in \mathbb{C}$.

(iv) Show that $e := (1, \wp, \wp')$ is a basis of $H^0(3p)$.

(v) Show that any choice of a basis of $H^0(3p)$ yields an embedding $E \to \mathbb{CP}^3$, which in fact can be factorized as $E \to \mathbb{CP}^2 \to \mathbb{CP}^3$, where the last map is a natural embedding.

Check also that the map $E \to \mathbb{CP}^2$ obtained in (v) realizes E as a plane algebraic curve $\{y^2 = 4x^3 + g_2 x + g_3\}$.

Return: Friday, 2019/02/08, at 2 pm before the lecture.