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1. Gauss Integral

Def 1 $s \in C^1(\mathbb{R})$ s.t. $|s(x)| \to \infty$ as $|x| \to \infty$

$Z(s) := \{ s(x) = 0 \}$, $s'(x) \neq 0 \quad \forall x \in Z(s)$

Gauss integral

$$Z = \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} s'(x) e^{-\frac{1}{2} s(x)^2}$$

Lemma $Z = \sum_{z \in Z(s)} \operatorname{sign} s'(z)$

Proof

$$\int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} s'(x) e^{-\frac{1}{2} s(x)^2} \quad \underline{\underline{y = \sqrt{\hbar}\, s(x)}}$$

$$= \int_{\sqrt{\hbar} s(-\infty)}^{\sqrt{\hbar} s(+\infty)} \frac{dy}{\sqrt{2\pi\hbar}} e^{-\frac{1}{2\hbar} y^2}$$

$$= \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi\hbar}} s' e^{\frac{1}{2\hbar} s(x)^2} = \int_{-\infty}^{+\infty} s'(x) \underbrace{\frac{e^{\frac{1}{2\hbar} s(x)^2}}{\sqrt{2\pi\hbar}}}_{\downarrow \, \delta} dx$$

$$= \sum_{z \in Z(s)} \frac{s'(z)}{|s(z)|}$$

$\boxtimes$

Def 3 Eucl. n-N general. for

$s \in C^1(\mathbb{R}^n)$

$$Z = \int_{\mathbb{R}} \prod_{i=1}^{n} \frac{dx^i}{(2\pi)^{n/2}} \det\left(\frac{\partial s^j}{\partial x^k}\right) e^{-\frac{1}{2} s(x)^2}$$

$$= \sum_{z \in Z(s)} \operatorname{sign} \det\left(\frac{\partial s^j}{\partial x^k}\right)$$

1. Z Counts solution for $s^*(x) = 0$ with a prefact.
2. Z is an oriented intersection number

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2. Superspace

- Superalgebra is a vect. space $A = A^0 \oplus A^1$ with a supercomm. multiplication, i.e.

$$f \in A^a,\ g \in A^b \implies f \cdot g \in A^{a+b}, \quad fg = (-1)^{a+b} gf$$

- Superspace $\mathbb{R}^{n|k}$ is defined by S.A. of coord functions

$$\left(C^\infty(\Omega)[\psi^1, \dots, \psi^k], +, \cdot\right) \text{ for open } \Omega \subset \mathbb{R}^n$$

$$\overset{\Downarrow}{\Phi} = \sum_{k=0}^{n} \sum_{i_1 < \dots < i_k} \phi_{i_1 \dots i_k} \psi^{i_1} \dots \psi^{i_k}$$

- For an n-dim mfld M the supermfld $\hat{M}^{(k)}$ is given by the sheaf of SA $\mathcal{O}_{\hat{M}^{(k)}}$ on M.

For open $U \subset M$, $\Omega \subset \mathbb{R}^n$ with chart

$U \xrightarrow{\cong} \Omega$ we have $\mathcal{O}_{\hat{M}^{(k)}}(U) \simeq C^\infty(\Omega)[\psi^1, .., \psi^k]$

<u>Construction 5</u> For a rk k-VB $E \to M$ we have in loc. trivialization $E|_U \cong U \times V$

$C^\infty(U) \otimes \Lambda^* V \cong \mathcal{O}_{\hat{M}^{(k)}}(U)$

For $E = TM$ (T^*M) we ~~have~~ make an identification $\psi^i \longleftrightarrow dx^i$ and

$$C^\infty(\hat{M}^{(k)}) \cong \Omega^*(M).$$

<u>Def. 6</u> "Ghost number" of a superfield $\mathcal{O}_\omega \in \hat{M}^{(k)}$

$$\deg \omega = gh \# \mathcal{O}_\omega \in \mathbb{Z} \qquad \Omega^*(M) \times (\Omega^*(M))^*$$

BRS-charge (coboundary op Q)

$$Q\,\mathcal{O}_\omega \longleftrightarrow d\omega$$

in loc coord.

$$Q x^i = \psi^i, \quad Q\psi^i = 0.$$

Integration of superfields via $\mathrm{ber}(x|\psi) \in \mathrm{Ber}(\Omega^*(\ldots$

$$\int_M \omega = \int_{\hat{M}} \mathrm{ber}(x|\psi)\,\mathcal{O}_\omega \text{ with } \cancel{\mathrm{ber}(x|\psi)} =$$

$$\mathrm{ber}(x|\psi) = dx^1 \ldots dx^n\, [d\psi^1 \ldots d\psi^n]$$

<u>Statement 7</u> We have $\int d\psi = 0$

and $\int [d\psi^1 \dots d\psi^n]\, \psi^1 \dots \psi^n = 1$.

3. <u>SUSY Integral</u>

<u>Lemma 8</u> For $M \in \mathrm{Mat}(n, \mathbb{R})$ $\exists$ a $\xi \in \{\pm 1\}$ s.t

$$\frac{\xi}{i^n} \int [\underbrace{d\overbrace{\psi^1}^{1}}_{-1} \dots d\psi^n \underbrace{d\overbrace{\chi_1}^{-1}}_{1} \dots d\chi]\, e^{i \sum_{jk} \chi_j M_{jk} \psi^k} = \det.$$

<u>Proof</u>

$$\frac{\xi}{i^n} \int [\ ] \sum_{\ell=0}^{\infty} \left(\frac{i^\ell}{\ell!} \left(\sum_{j,k}^{k} \chi_j M_{jk} \psi^k \right)^\ell \right) =$$

$$= \frac{\xi}{i^n} \int [\ ] \frac{i^n}{n!} \left(\sum_{j,k}^{n} \chi_j M_{jk} \psi^k \right)^n$$

$$= \xi \int [\ ] \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^{n} \chi_i \psi^{\sigma(i)} M_{i\sigma(i)}$$

$$\cancel{= \pm \xi \int [\ ] \frac{1}{n!} \sum_{\sigma \in S_n}}$$

$$= \pm \xi \underbrace{\int [\ ] \chi_n \dots \chi_1 \psi^1 \dots \psi^n}_{1} \underbrace{\sum_{\sigma \in S_n} \operatorname{sign} \sigma \prod_{i=1}^{n} M_{i\sigma(i)}}_{\det M}$$

Constr. 9

Recall $Z = \int_{\mathbb{R}^n} \frac{\prod dx^i}{(2\pi)^{n/2}} \underbrace{\det(\;)}_{\text{replace with the exp. in the Lemma}} e^{\cdots}$

$$Z = \frac{\xi}{i} \int_{\hat{\mathbb{R}}^n} \int_{\cancel{(\hat{\mathbb{R}})^*}\; \Pi(\mathbb{R}^n)^*} \frac{\mathrm{ber}(x \mid \psi, \chi)}{(2\pi\hbar)^{n/2}} e^{-\frac{1}{2\hbar} S^j(x) S^j(x) + i \chi_j \frac{\partial S^j}{\partial x_i}}$$

Want to make this Q-invariant.

$gh\#\, \psi^i = 1$, $gh\#(\chi_a) = -1$

Need auxiliary field H_a with $gh\#(H_a) = 0$

$x_0 \in \mathbb{R}^n$ $\quad f(H_a) = H_a + i \frac{S^a(x_0)}{\hbar}$ has one solution

$$\int_{\mathbb{R}^n} \left(\frac{\hbar}{2\pi}\right)^{n/2} \prod_{a=1}^{n} dH_a\, e^{-\frac{\hbar}{2}\left(H_a + \frac{i S^a(x_0)}{\hbar}\right)^2} = \pm 1$$

$$Z = \frac{\xi}{(2\pi i)^n} \int_{\hat{\mathbb{R}}^n \times (\hat{\mathbb{R}}^n)^*} \mathrm{ber}(X, H \mid \psi, \chi)\, e^{-\frac{\hbar}{2} H_j H_j - i H_j S^j + i \chi_j \frac{\partial S_j}{\partial x^k}}$$

$$S = \frac{\hbar}{2} H_j H_j + i H_j S^j - i \chi_j \frac{\partial S^j}{\partial x_k} \psi^k$$

(H_a, χ_a) antighost multiplet

Define $Q \chi_a = H_a$, $\quad Q H_a = 0$

$-S = Q(\Psi)$ for

$$\Psi = -\frac{\hbar}{2}\chi_a H_a - i\chi_a s^a$$

Z is invariant under the change

$$S \longrightarrow S - Q(\underbrace{\Delta\Psi}_{\text{variation of } \Psi})$$

(provided the behaviour at ∞ is not changed)

"Z localizes to $Z(s)$" $\Leftrightarrow$ ρ is supported at $-S=0$

$\Leftrightarrow$ supp is at $Q(\Psi)=0$ $\Leftrightarrow$

"Z localises at Q-Fixedpoints".

4. <u>SUSY correlation functions</u>

<u>Constr. 11</u> Define a superfield

$$\widehat{Eul}_n : \hat{\mathbb{R}}^n \longrightarrow \hat{\mathbb{R}}^n \quad (\text{rather } \widehat{Eul}_n \in C^\infty(\hat{\mathbb{R}}^n)?)$$

$$\widehat{Eul}_s := \int_{\hat{\mathbb{R}}^n} \frac{dH_1 \dots dH_n [d\chi_1 \dots d\chi_m]}{(2\pi i)^m} e^{Q(\Psi)}$$

with $s: \mathbb{R}^n \longrightarrow V \cong \mathbb{R}^m$, $m \in \mathbb{N}$

- $gh \# \widehat{Eul}_s = m$

$$Q(\widehat{Eul}_s) = 0 \implies \exists\ Eul_s \in \Omega^m(\mathbb{R}^n)$$
s.t. $d(Eul_s) = 0$

Constr. 12

For $O_\omega : \widehat{\mathbb{R}^n} \longrightarrow \underset{\widehat{\mathbb{R}^n} \times \widehat{(\mathbb{R}^n)^*}}{\cancel{\mathbb{E}}}$ with $Q O_\omega = 0$

$$\langle O_\omega \rangle = \int_{\widehat{\mathbb{R}^n}} \mathrm{ber}(x \mid \psi)\, O_\omega\, \widehat{\mathrm{Eul}_s}$$

- If $Z \neq 0 \Leftarrow gh\#(O_\omega) = n-m$

- $\langle O_\omega \rangle$ depends on the cohomology class only
- Localization identity

$$\langle O_\omega \rangle = \int_{\widehat{\mathbb{R}^n}} \mathrm{ber}(x \mid \psi)\, O_\omega\, \widehat{\mathrm{Eul}_s} = \int_{\mathbb{R}^n} \omega \wedge \mathrm{Eul}_s$$

$$= \int_{Z(s)} i^* \omega$$

Talk 2 Evgenij Pascual

$$\langle O_\omega \rangle = \int_{\widehat{\mathbb{R}^n}} \mathrm{ber}(x, \psi)\, O_\omega\, \widehat{\mathrm{Eul}_s}$$

$$= \int_{\mathbb{R}^n} \omega \wedge \mathrm{Eul}_s = \int_{Z(s)} i^* \omega$$

Thom isomorphism

Poincaré lemma

$$H^*(M\times\mathbb{R}^n) = H^*(M)$$

$$H_c^*(M\times\mathbb{R}^n) = H_c^{*-n}(M)$$

For any VB $H^*(E) \cong H^{*-n}(M)$, $n = \text{rk}\, E$

but in general $H_c^*(E) \not\cong H_c^{*-n}(M)$

However, if M, E are orientable, then

$$H_c^*(E) \cong H_c^{*-n}(M)$$

Pf

$$H_c^*(E) \underset{\text{PD}}{\cong} \left(H^{m+n-*}(E)\right)^* \cong \left(H^{m+n-*}(M)\right)^* \underset{\text{PD}}{\cong}$$

$$\cong H_c^{*-n}(M).$$

$\blacksquare$

Compact support in vertical direction

$\pi_*: \Omega_{cv}^*(E) \longrightarrow \Omega^{*-n}(M)$ integration along fibers

Projection formula

τ form on M, ω form on E, $\omega \in \Omega_{cv}(E)$

Then $\pi_*(\pi^*\tau\wedge\omega) = \tau\wedge\pi_*\omega$

Prop If E is oriented, then

$$\begin{tikzcd} \pi_{cv}^* : H^*_{cv}(E) \arrow[r, "\cong"] & H^{*-n}(M) \arrow[l, bend left, "\tau"] \end{tikzcd}$$

Want to find π_*^{-1} (the Thom iso)

Consider $H^0(M) \ni 1$ has a well-defined image in $H^n_{cv}(E)$, which we call Φ, the Thom class of E.

$$\pi_*(\pi^*\omega \wedge \Phi) = \omega \wedge \pi_* \Phi = \omega.$$

$$\Downarrow$$

$$\tau(\cdot) = \pi^*(\cdot) \wedge \Phi$$

Also important: Poincaré duality

Let S be a closed oriented submfld of M, then its Poincaré dual η_S is determined by

$$\int_S \omega = \int_M \omega \wedge \eta_S \quad \forall \omega \in \Omega_c^*(M)$$

Prop η_S is the same as Φ of the normal subbundle of S.

Euler class $E \xrightarrow{\pi} M$, $s_0 : M \to E$ zero section

$\Gamma(E) \ni s \pitchfork s_0 =: I$. Then

$\dim I = \dim M - r$, $r = \operatorname{rk} E$

and we set $e_s := PD(I)$

Now SUSY

For $\pi: E \to M$, E orientable, have the Thom iso

$$H^i(M) \cong H^{i+m}_{cv}(E)$$

Also, $s^*\Phi(E) = e_s$

$$\int_M \omega \wedge s^*\Phi(E) = \int_{Z(s)} \iota^*\omega$$

So need to replace $\frac{\partial s}{\partial x_j}$ by $\nabla_j s$

Let $\{e^a\}$ be an ON basis of E, then define ∇ by $\nabla e^a = dx^j \theta_j^{ab} e_b$

Then $S = \frac{\hbar}{2} H_j H_j + i H_j s^j - i \chi_j \frac{\partial s^j}{\partial x_k} \psi^k$

~~covariantize~~ $\cancel{S = \frac{\hbar}{4}}$

$$S = -\frac{1}{2\hbar} s^j s^j + i \chi_j \frac{\partial s^j}{\partial x_k} \psi^k$$

covariantize $S(x,\nabla) = -\frac{1}{2\hbar} s^a s^a + i \chi_a (\nabla_j s)^a \psi^j +$

$$+ \frac{\hbar}{4} \chi_a \chi_b F_{ij}^{ab} \psi^i \psi^j$$

This form is obtained from the requirement $S = Q(\Psi)$, where Ψ is covariantized

Define a superfield on $\hat{M}$

$$\widehat{\mathrm{Eul}}_s(E,\nabla) := \int \prod_{a=1}^{n} \frac{d\chi_a dH_a}{2\pi i} e^{S}$$

Have lin. op. $\nabla S : T_p M \to E_p$

Important that $Z(s)$ is a mfld.
This is the case if ∇ is not surjective.

How to get a localization formula for $\hat{E}$?

Consider ex. seq.

$$0 \to \mathrm{Im}\,\nabla S \to E \to \underset{\underset{E/\mathrm{Im}\,\nabla S}{\shortparallel}}{\mathrm{Cok}\,\nabla S} \to 0$$

Coker ∇S

Then

$$\int_{\hat{E}} \mathrm{ber}(X, H \mid \Psi, X)\, e^{S} O = \int_{\hat{M}} \mathrm{ber}(X \mid \Psi)\, \widehat{\mathrm{Eul}}_s(E,\nabla)$$

$$= \int_{Z(s)} i^* \omega_0 \wedge \mathrm{Eul}(\mathrm{Coker}\,\nabla S)$$

Here we require that Coker(∇S) is a bundle

Need to replace $H^*_{cv}(E) \longrightarrow H^*_{rd}(E)$
rapidly decaying

Talk 3 Victor Pidstrygach

Equivariant cohomology

Algebra A $\quad \mathrm{Der}\, A = \{ \delta : A \to A$ linear $\mid$
$\delta(f \cdot g) = \delta(f) \cdot g + f\,\delta(g) \}$

For an A-module F

$\mathrm{Der}\,(A, F) = \{ \delta : A \to F$ linear $\mid \delta(fg) = \delta(f) \cdot g + f\delta(g) \}$

Exercise $[\delta_1, \delta_2] \in \mathrm{Der}\,(A, F)$

Ex $\quad A = C^\infty(\mathbb{R}^n)$

$$\mathrm{Der}\,(A) = \left\{ \sum a_i \frac{\partial}{\partial x_i} \;\middle|\; a_i \in C^\infty(\mathbb{R}^n) \right\}$$

$$\mathbb{R}^n \ni x \rightsquigarrow m_x = \{ f \in C^\infty(\mathbb{R}^n) \mid f(x) = 0 \}$$

$$C^\infty(\mathbb{R}^n)/m_x \cong \mathbb{R} \text{ field}$$

$$T_x^\vee \mathbb{R}^n = m_x/m_x^2, \quad T_x \mathbb{R}^n = (m_x/m_x^2)^\vee$$

$$= \mathrm{Der}\left(C^\infty(A), C^\infty(\mathbb{R}^n)/m_x\right)$$

$C^\infty(\Pi T^* M) = \Omega^\bullet(M)$ graded algebra, superalgebra

$$\alpha \wedge \beta = (-1)^{|\alpha||\beta|} \beta \wedge \alpha$$

d_{dR}, X vector field $\quad \mathcal{L}_X$ Lie derivative

ι_X interior derivative

$\deg d_{dR} = 1$

$\deg \alpha_X = 0$ $\qquad \deg \iota_X = -1.$

$$[\alpha_X, \alpha_Y] = \alpha_{[X,Y]} = \alpha_X \alpha_Y - \alpha_Y \alpha_X$$

$$[\alpha_X, \iota_Y] = \iota_{[X,Y]}$$

$$[\iota_X, \iota_Y] = \iota_X \iota_Y + \iota_Y \iota_X = 0$$

$$[d, \alpha_X] = 0$$

$$[d, d] = d \circ d + d \circ d = 0$$

$$\alpha_X = [d, \iota_X] = d\,\iota_X + \iota_X\, d$$

Lie gp G acts on M

$\mathfrak{g} = \mathrm{Lie}(G) \ni \xi \longmapsto K^\xi$ fundamental vector field

$$K^\xi(p) = \frac{d}{dt}\Big|_{t=0} \left(e^{-t\xi} \cdot p \right)$$

$$\iota_{K^\xi}, \quad \alpha_{K^\xi}, \quad d$$

$\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ supersymmetry of geometric symmetry

$$C(\mathrm{spt}) := \mathbb{R}[\theta]/\theta^2 = \Lambda^1 \mathbb{R}^1$$

$$\Pi TM = \mathrm{Map}(\mathrm{spt}, M) = \mathrm{Map}\left(C^\infty(M), \mathbb{R}[\theta]/\theta^2 \right).$$

Ex $M = \mathfrak{g}$, G acts by the adj. action

$\Rightarrow$ fundamental vector fields

symmetries α_{K^ξ}, ι_{K^ξ}, d_{dR} of $\Pi T\mathfrak{g}$

$C^\infty(\Pi T^*\mathfrak{g}) = \Omega^\bullet(\mathfrak{g}) \supset \{$diff. forms with polynomial coeffic.$\}$

$\qquad = S^\bullet\mathfrak{g}^\vee \otimes \Lambda^\bullet \mathfrak{g}^\vee$

is generated by

$l \otimes 1$, $1 \otimes l$ $\qquad l \in \mathfrak{g}^\vee$

degree 1 diff. forms with constant coefficients

$$d_{dR}(l \otimes 1) = 1 \otimes l \quad , \quad d_{dR}(1 \otimes l) = 0$$

$$K^\xi_\eta = [\xi, \eta] = ad_\xi(\eta)$$

~~$\alpha_{K^\xi}(1 \otimes l)(\eta) =$~~

$$\alpha_{K^\xi}(l \otimes 1) = l \circ ad_\xi \otimes 1 \quad *$$

$$\alpha_{K^\xi}(1 \otimes l) = 1 \otimes l \circ ad_\xi$$

We also define another derivation d_K

$$d_K(1 \otimes l) = l \otimes 1 \qquad d_K(l \otimes 1) = 0$$

$$[d_K, d_{dR}]\Big|_{S^k\mathfrak{g}^\vee \otimes \Lambda^l \mathfrak{g}^\vee} = (k+l)\,1$$

$k + l = 0 \Leftrightarrow S^0\mathfrak{g}^\vee \otimes \Lambda^0 \mathfrak{g}^\vee = \mathbb{R}$

$$H^i(\mathcal{K}(\mathfrak{g}), d_K) = \begin{cases} \mathbb{R} & i=0 \\ 0 & i>0. \end{cases}$$

$$\hat{\iota}_{K^\xi}(\ell \otimes \mathbb{1}) = \hat{\iota}_{K^\xi} d_K (1 \otimes \ell) = \mathcal{L}_{K^\xi}(1 \otimes \ell) - d_K \hat{\iota}_{K^\xi}(1\otimes$$

$$= 1 \otimes \ell \circ \mathrm{ad}_\xi \qquad (**)$$

This defines a nonstandard ~~sy~~ supersymmetry.

G acts freely on a mfld P s.t. $P \to P/G = M$ is a principal bundle

Connection 1-form $a \in \Omega^1(P; \mathfrak{g})^G$ $\quad R_g^* a = \mathrm{Ad}_{g^{-1}}(a)$

$$a(K^\xi) = \xi$$

$$\Omega^\bullet(P; V)_{hor} = \{ \alpha \in \Omega(P, V) \mid \iota_{K^\xi} \alpha = 0 \ \ \forall \xi \in \mathfrak{g} \}$$

$$\Omega(P; V)^G = \{ \alpha \in \Omega(P, V) \mid R_g^* \alpha = \rho_g(\alpha) \}$$

Here V is any rep. of G

$$\Omega(P, V)_{bas} = \Omega(P, V)^G_{hor} \overset{\text{Claim}}{=\!=} \Omega^1(M, E)$$

$$E = P \times_G V$$

Topological idea of equivar. cohomology,

1. find a contr. top. space EG with a free action of G
2. G-action on $N \times EG$ is free, so compute

$$H^\bullet(N \times EG/G) = H^\bullet_G(N)$$

~~Def Let A be a superalgebra~~

$\hat{\mathfrak{g}} := \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \langle d \rangle$

deg $\quad -1 \quad 0 \quad +1$

Def Let A be a superalgebra with $\hat{\mathfrak{g}} \hookrightarrow \mathrm{Der}(A)$

We say the pair has property C if

$\exists$ elements $a_1, \dots, a_n$ s.t. $\hat{\iota}_{\xi_i} a_j = \delta_{ij}$ $\quad (*)$

deg 0

Ex $\mathcal{K}_{\mathfrak{g}}$ - Koszul, $(**)$ implies the pair has property C

Def $H_G(A) = H(A \otimes W(\mathfrak{g})_{bas}, \hat{d} + d_K)$

basic means $\left(\bigcap_{\xi \in \mathfrak{g}} \operatorname{Ker} \iota_\xi\right) \cap \left(\bigcap_{\xi \in \mathfrak{g}} \operatorname{Ker} L_\xi\right)$

Weil algebra model

Claim $H(A \otimes W(\mathfrak{g})_{bas}, \hat{d} + d_K) = H(A \otimes S^\bullet(\mathfrak{g}^\vee)^G, \hat{d} + \iota_{\mathfrak{g}})$

Mathai-Quillen isomorphism

$A = \Omega^\bullet(M)$ with a G-action and standard

$\iota_{K^\xi}, \mathcal{L}_{K^\xi}, d_{dR}$

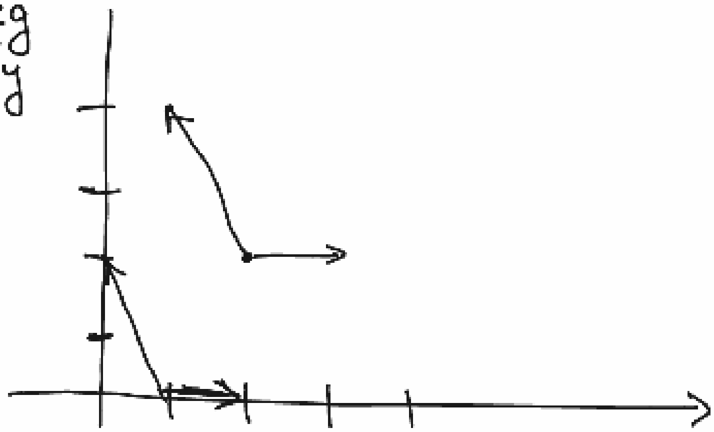
$S^\bullet(\mathfrak{g}^\vee) \otimes \Omega^\bullet(M) = \mathrm{Hom}(S\mathfrak{g}, \Omega^\bullet(M))$

$\ni \alpha$ $\qquad \alpha \in S^d(\mathfrak{g}^\vee) \otimes \Omega^k(M)$

~~$(d\alpha)[\xi] := d(\alpha[\xi])$~~

$$\Big(\big(\underbrace{d - \iota_y}_{d_y}\big)\underset{d}{\alpha}\Big)\underset{k}{[\xi]} = d\big(\alpha[\xi]\big) - \iota_{K\xi}\big(\alpha[\xi]\big)$$

form degree $d+1$, poly degree k form deg $d-1$, poly deg $k+1$



$$\deg \alpha = \text{deg form} + 2\,\text{poly degree}$$

$$(d_{dR} - \iota_{K\xi})^2 = \underset{\overset{\shortparallel}{0}}{d_{dR}^2} - [d, \iota_{K\xi}] + \underset{\overset{\shortparallel}{0}}{(\iota_{K\xi})^2} = -\alpha_{K\xi}$$

α is invariant $\Rightarrow \alpha_{K\xi} = 0 \Rightarrow (d_{dR} - \iota_{K\xi})^2 = 0$

<u>Interpretation</u> $\alpha \in S^\bullet(y^\vee) \otimes \Omega(M)$

M cmpt $\int_M \alpha \in S^\bullet(y^\vee)$

$\alpha = d_y \beta \Rightarrow \int_M \alpha = 0$

$\alpha_{[d]}$ d-form component of α

$d_y \alpha = 0 \iff \alpha_{[n]} = d\beta_{[n-1]} - i_{K\xi}(\underset{\overset{\shortparallel}{0}}{\cancel{\beta_{[n+1]}}})$

G acts on M, $\alpha \in S^\bullet(\mathfrak{g}^\vee) \otimes \Omega^\bullet(M)$

$d_\mathfrak{g} \alpha = 0$

M_0 = zeros of K^ξ.

<u>Lemma</u> $\alpha(\xi)$ is exact on $M \setminus M_0$

<u>Pf</u> Define 1-form $\Theta(X) = g(K^\xi, X)$

$$d_{K^\xi} \Theta = \underbrace{|K^\xi|^2}_{\deg 0} + \underbrace{d\Theta}_{\deg 2}$$

On $M \setminus M_0$ we have $\alpha(\xi) = d_{K^\xi}\left(\frac{\Theta \wedge \alpha(\xi)}{d_{K^\xi}\Theta(\xi)} \right)$

since $\left(|K^\xi|^2 + d\Theta\right)^{-1} = \frac{1}{|K^\xi|^2}\left(1 - |K^\xi|^2 d\Theta + \ldots\right)$

$$\Rightarrow \quad \alpha(\xi)_{[n]} = d\left(\quad \right)_{[n-1]}$$

$\square$

<u>Thm</u> Assume K^ξ has isolated zeros $Z(K^\xi) = \{p_0, \ldots, p_n\}$. Then

$$\int_M \alpha(\xi) = (-2\pi)^p \sum \frac{\alpha(\xi)(p)}{\sqrt{\det(L_p)}}$$

where L_p - linearisation of K^ξ at p.

Poddsrygach Equiv. Cohomology II
11.05.18 Göttingen

$$\underbrace{\Lambda^\bullet \mathfrak{g}^\vee \otimes S^\bullet(\mathfrak{g}^\vee)}_{W^\bullet(\mathfrak{g})} \otimes \mathfrak{g} \supset \mathfrak{g}^\vee \otimes 1 \otimes \mathfrak{g} \ni id = a^{\mathfrak{K}} \text{ connection}$$

$$\supset \Lambda^2 \mathfrak{g}^\vee \otimes 1 \otimes \mathfrak{g} \ni [\cdot,\cdot] = F_{a^{\mathfrak{K}}} \text{ curvature}$$

Mathai-Quillen iso:

$$\exp(\iota^M_{K a^{\mathfrak{K}}}) : \Omega(M) \otimes W(\mathfrak{g})_{hor} \longrightarrow \Omega(M) \otimes W(\mathfrak{g})_{hor}$$

and $d_{dR} + d^{\mathfrak{K}}$ to $d_{dR} - \iota^M_{K F_{a^{\mathfrak{K}}}} + \alpha_{K a^{\mathfrak{K}}}$

On $\left(\Omega(M) \otimes W(\mathfrak{g})\right)^G_{hor} \longrightarrow \left(\Omega(M) \otimes W(\mathfrak{g})_{bas}\right)^G$

$$d_{dR} + d^{\mathfrak{K}} \rightsquigarrow d_{dR} - \iota^M_{K a^{\mathfrak{K}}} := d_G$$

Want to have a formula

$$\int_Y \omega = \int_P \overset{= \pi^* \omega}{\tilde{\omega}} \cdot P(P \to Y) \qquad G \to P \to Y$$

$$\alpha \in \left(\Omega^\bullet(M) \otimes S^\bullet \mathfrak{g}^\vee\right)^G \qquad \int_P \alpha \in S^\bullet(\mathfrak{g}^\vee)$$

not a number so can't be compared to $\int_Y \omega$ easily

$$\oint_{\varepsilon, P} \alpha := \frac{1}{\operatorname{Vol} G} \int_{\mathfrak{g}} d\xi \, \overbrace{e^{-\frac{\varepsilon}{2}|\xi|^2}} \int_P \alpha$$

~~$d_G \beta = \oint_P \alpha = 0$~~

Extend this integral to an integral over $P \times y \times W(y)^\vee$

$P(P\to Y) = \int_{W(y)^\vee} e^{Q(\Psi_{proj})} d\bar{\xi}\, dy$ (Witten's answer)

$$\Psi_{proj} = i\langle \xi, a\rangle + \langle \bar{\xi}, [\xi, \eta]\rangle$$

$$W(y) = \Lambda^\bullet y^\vee \otimes S^\bullet y^\vee \qquad W(y)^\vee = \underset{-2}{S^\bullet y^\vee} \otimes \underset{-1}{\Lambda^\bullet y^\vee}$$

$$Q(\ell\otimes 1) = 1\otimes \ell \quad \equiv \quad Q\bar{\xi} = \eta$$

$$Q(1\otimes \ell) = \mathcal{L}^{K}(\ell\otimes 1) \equiv Q\eta = -\mathcal{L}_{K^\xi}\bar{\xi}$$

$$\int_Y \omega = \int_P \pi^*\omega\, P(P\to Y) = \int_{\substack{y\times W(y)^\vee\times P \\ \xi \in \quad \bar{\xi},\eta \in}} \pi^*\omega\cdot e^{Q(\Psi_{proj}) - \frac{\varepsilon}{2}\langle\bar{\xi}, d\bar{\xi}\rangle}$$

Emanuel Scheidegger Topological twist

4-mfld, G-bundle w/ conn

$$YM(A) = \int_M tr(F_A \wedge * F_A) \qquad F = dA + [A,A]$$

In physics: YM theory

add SUSY $\longrightarrow$ Super-Yang-Mills theory

in various ways

$N=1 \rightsquigarrow$ Kähler

$N=2 \rightsquigarrow$ Donaldson invariants, SW inv

$N=4 \rightsquigarrow$ Kapustin-Witten, geom. Langlands

To make mathematically well-defined:

top. twisted Super-Yang-Mills thy is an example of $\begin{cases} \text{top. field theory of Witten's type (phy)} \\ \text{coh. field theory (math)} \end{cases}$

Basic ingredients

- G cmpt Lie gp, $\mathfrak{g} = \mathrm{Lie}(G)$
  gauge gp in physics
- (X, g) closed oriented 4-mfld
- $P \to X$ principal G-bundle

$M \rightsquigarrow \mathcal{A} := \mathrm{Conn}(P)$.

$E \to M \rightsquigarrow \mathcal{E} := \mathcal{A} \times \Omega^{2,+}(X, \mathrm{ad}\, P)$, $\quad s$

$s(A) := F_A^+$

$G \rightsquigarrow \mathcal{G} = \mathrm{Aut}(P)$ (gauge gp in math)

X
ψ
χ
H $\rightsquigarrow$?
ϕ
$\bar{\phi}$
η

$N=2$ QFT in 4D

fields = (distributional) representations of the $\boxed{N=2 \text{ Super-Poincaré alg.}}$ (on vector bundles over X)

A non-ass super-alg ($\mathbb{Z}_2$-graded) over a comm. ring R is a Lie superalgebra, if $\exists\ [\cdot,\cdot] : A \times A \to A$

s.t. $[X,Y] + (-1)^{|X||Y|}[Y,X] = 0$

$(-1)^{|Z||Y|}[Z,[X,Y]] + (-1)^{|X||Z|}[X,[Y,Z]] + (-1)^{|Y||X|}[Y,[Z,X]] =$

$L = L_0 \oplus L_1 \Rightarrow L_0$ is a Lie alg.

If $\exists$ at least one odd element, then

L_1 is an L_0-module for ad $X \mapsto [Y, X]$

If $\exists$ at least two odd elements, then

$\exists$ L_0-equivariant symm map

$\{\cdot,\cdot\} : L_1 \otimes L_1 \to L_0$ s.t.

$$[\{X,Y\}, Z] + [\{Y,Z\}, X] + [\{Z,X\}, Y] = 0$$
$$\forall X, Y, Z \in L_1.$$

V real quadr. v. space (of dim 4)

$L_0' = V \rtimes \underline{so}(V)$ Poincaré alg.

$L_0 := L_0' \oplus R$, R any reductive Lie alg. (R-symme

(reductive: adj. repr. is completely reducible $\Leftrightarrow$
$= s + a$, s semisimple & a abelian)

$R = \underline{u}(1)$ $N = 1$ extended super Poincaré-algebra

$R = \underline{su}(2) \oplus \underline{u}(1)$ $N = 2$ —— // —— // —— // ——

S real spinorial rep. of $Spin(V)$

$\exists$ symm. nonzero map $\Gamma : S \times S \to V$, which is
equiv. wrt $Spin(V)$

S is an L_0-module by requiring that
V acts as 0 on S, and is a rep. of R.

$L_1 := S$ with $[s_1, s_2] := \Gamma(s_1, s_2)$ Neo smartpen

<u>Special case</u> $\dim V = 4$ (didn't use this above)

$$L_0 = \mathbb{R}^4 \rtimes (\mathfrak{su}(2)_- \oplus \mathfrak{su}(2)_+) \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R$$

$$L_1 = (1,2,2)^1 \oplus (2,1,2)^{-1}$$

$$(1,2,2) = \underset{\substack{\mathfrak{su}(2)_- \\ \dim = 1}}{V_-} \otimes \underset{\substack{\mathfrak{su}(2)_+ \\ \dim = 2}}{V_+} \otimes \underset{\substack{\mathfrak{su}(2)_R \\ \dim = 2}}{V_R}$$

$(\ \ )^1$ weight of the $\mathfrak{u}(1)$ repres.

There is an L_0 equiv. map $\{\cdot,\cdot\} : \mathrm{Sym}^2(L_1) \to \mathbb{R}^4$

Physics notations

Basis elements in $(1,2,2)^1 : \ni \bar{Q}_{\dot\alpha}^A$ $\quad \dot\alpha = 1,2$; $A = 1,2$

$(2,1,2)^{-1} \ni Q_\alpha^A$ $\quad \alpha = 1,2$; $A = 1,2$

$$\{Q_\alpha^A, \bar{Q}_{\dot\alpha}^B\} = 2\,\varepsilon^{AB}\,\sigma^\mu_{\alpha\beta} P_\mu \quad , \mu = 1,2,3,4$$

~~totally~~ antisymm. tensor

$$\varepsilon^{A_1 \dots A_n} = \begin{cases} +1, & \text{if } (A_1,\dots,A_n) = (1,\dots,N) \\ -1, & \text{if } (A_1,\dots,A_n) = (2,1,\dots,N) \\ 0, & \text{if } A_i = A_j \text{ for some } i \neq j. \end{cases}$$

$\sigma^\mu_{\alpha\beta}$ is the repr. matrix of the map Γ

= Pauli matrices.

Q's are called supercharges

P's are called momenta

Also, $\{Q^A_\alpha, Q^B_\beta\} = 0$.

$\mathcal{N}=2$ vector multiplet

Reps of Lie superalgebras are determined by corr. reps of their even parts. Consider the following rep. of 𝔤

Basis elements	$\underline{su}(2)_- \oplus \underline{su}(2)_+ \oplus \underline{su}(2)_R$	$u(1)_R$
A_μ	$(2,2,1)$ $\oplus$	0
$\overline{\Psi}^A_{\dot\alpha}$	$(1,2,2)$ $\oplus$	-1
Ψ^A_α	$(2,1,2)$ $\oplus$	$+1$
ϕ	$(1,1,1)$ $\oplus$	2
$\overline{\phi}$	$(1,1,1)$ $\oplus$	-2
D^a	$(1,1,3)$	0

$\mu = 1,2,3,4$

$A = 1,2$

$a = 1,2,3$

$\uparrow$ ghost numbers (later)

D auxiliary field; appears only algebraically in the EL equs; can be eliminated

<u>Globalize</u> $\mathbb{R}^4 \rightsquigarrow X^4$

vector spaces $\rightsquigarrow$ vector bundles

We must introduce a new datum, the R-symmetry bundle $P_R \to X$ a principal $SU(2)_R$-bundle.

$\Rightarrow A$ is a con-n.

$\phi, \bar\phi$ sections of $\mathrm{ad}P \otimes \mathbb{C}$

D —— // —— $\mathrm{ad}P \otimes W$ $\Leftarrow$ real rk $= 3$ v.b. ass. to P_R

$$\Psi, \bar\Psi \in \Gamma(\underset{\substack{\nearrow\\ \text{spinor}\\ \text{bundles}\\ \text{of } X}}{S^\pm} \otimes \underset{\substack{\backslash\\ \text{spinor}\\ \text{rep}\\ \text{of } SU(2)_R}}{S_R} \otimes \mathrm{ad}P)$$

X does not need to be Spin!

In this case: $P_R \to X$ is an $SO(3)$ principal bundle s.t. $w_2(P_R) = w_2(X)$

Then $S^\pm \otimes S_R$ exist but $S^\pm$ and S_R don't.

Moore: Choose $P_R \cong P_+$ s.t. $P_+ \times_{SO(3)_+} \mathbb{R}^3 \cong \Lambda^2_+ T^* X$.

<u>Action</u>

Levi-Civita $\quad \omega \in \Omega^1(P_{SO(4)}, \underset{\substack{\shortparallel \\ so(3)_+ \oplus su(2)_-}}{so(4)})$

$$\omega = \omega_+ + \omega_-$$

Path integral

$$Z(\omega_+, \omega_-, \cancel{\omega_R}) := \int [\mathcal{D}A\, \mathcal{D}\Psi \dots \mathcal{D}D]\, e^{-S^{YM}_{phys}} \quad (\ )$$

$$S^{YM}_{phys} := -\int_X \frac{1}{g_0^2} \mathrm{tr}\left(F \wedge *F\right) + D\phi \wedge * D\phi^* - \frac{1}{4}[\phi, \phi^*]^2 \mathrm{vol}$$

bosonic part, depends on g

$$+ \underbrace{\frac{\theta_0}{8\pi^2} \int_X \mathrm{tr}(F \wedge F)}_{\text{topological term}} + \int_X \cancel{C \star (\bar\Psi \not{D} \Psi) + \bar\Psi \Psi \bar\Psi \dots}$$

$$+ \int_{X} \frac{1}{g^2 4\pi} \left(\overline{\Psi} \not{D} \Psi + \overline{\Psi} \Psi \overline{\Psi} \Psi + \overline{\phi} [\Psi, \Psi] + \right.$$

$$\left. + \phi [\Psi, \overline{\Psi}] \right)$$

The field D does not appear here, since it is integrated out

$$\mathrm{tr}(XY) = -\frac{1}{2h^\vee} \mathrm{tr}_{\mathfrak{g}} \left(\mathrm{ad}(X) \, \mathrm{ad}(Y) \right)$$

$\mathcal{G} = \mathrm{Aut}(P)$ (symmetries)

$$\mathcal{M} = \{ A \in \mathcal{A} \mid F_A^+ = 0 \} / \mathcal{G}$$

$N=2$ vector multiplet

$su(2)_+ \oplus su(2)_- \oplus su(2)_R \oplus u(1)_R$

A_μ,	$(2,2,1)$	ϕ	$(1,1,1)_2$
$\overline{\psi}^A_{\dot\alpha}$	$(1,2,2)_{-1}$	$\overline{\phi}$	$(1,1,1)_{-2}$
ψ^A_α	$(2,1,2)_{+1}$	D^a	$(1,1,0)_0$

Path integral

$$Z[\omega_+, \omega_-, \omega_R] = \int [\mathcal{D}A \mathcal{D}\psi \dots \mathcal{D}D]\, e^{-S^{SYM}_{phys}(A, \dots)}$$

$$S^{SYM}_{phys} := -\int_X \frac{1}{g_0^2} \mathrm{tr}\left(F_A \wedge * F_A\right) + D\phi \wedge * D\phi^*$$
$$- \frac{1}{4} [\phi, \phi^*]\, \mathrm{vol}_g + \frac{\theta_0}{8\pi^2} \int_X \mathrm{tr}(F \wedge F)$$
$$+ \frac{1}{g_0^2} \int_X \mathrm{tr}\Big(\overline{\psi} D \psi + \psi \overline{\psi} \psi \overline{\psi} + \overline{\phi}[\psi, \psi] +$$
$$+ \phi [\psi, \overline{\psi}] \Big).$$

$$\mathrm{tr}(XY) = -\frac{1}{2h^\vee} \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$$

Observation: If $\omega^+ = \omega_R$ $(P^+ \cong P_R)$
then Z does not depend on ω_+, ω_- !

Witten's idea of the topological twist.

In fact: Choose a new Lorentz subalgebra
(instead of $\mathfrak{su}(2)_- \oplus \mathfrak{su}(2)_+$)

$$\mathfrak{su}(2)_- \oplus \underline{\mathfrak{su}}(2)_+ ,$$

where $\underline{\mathfrak{su}}(2)_+ \underset{\text{diagonally}}{\subset} \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_R$

A_μ $(2,2)_0$ $\qquad \phi$ $(1,1)_2$

$\boxed{\bar{\psi}^A_{\dot\alpha} \ (1, 2\otimes 2)_{-1}}$ $\qquad \bar\phi$ $(1,1)_{-2}$

ψ^A_α $(2,2)_{+1}$ $\qquad D^a$ $(1,3)_0$

ψ_μ

Since $2\otimes 2$ is not irreducible, $\bar\psi^A_{\dot\alpha}$ decompose

$$(1, 2\otimes 2)_{-1} \Big\langle \begin{matrix} (1,1)_{-1} & \eta \\ (1,3)_{-1} & \chi_{\mu\nu} = -\chi \end{matrix}$$

self dual 2-form

(here we use $3 \cong \mathfrak{su}(2)\otimes\mathbb{C}$, $\mathfrak{su}(2) = \mathfrak{so}(3)$)

$Q^A_{\dot\alpha} \in (1,2,2)_{+1} \overset{\text{twist}}{\rightsquigarrow} (1, 2\otimes 2)_{+1} = (1,1)_{+1} \oplus (1,3)_1$

$Q^A_\alpha \in (2,1,1)_{-1} \overset{\text{twist}}{\rightsquigarrow} (2,1)_{-1}$ $\qquad Q := \delta^{\dot\alpha}_A \bar{Q}^A_{\dot\alpha}$

Q is the projection of $Q^A_{\dot\alpha}$ to $\mathbf{1}$.

$\{\bar{Q}^A_\alpha, Q^B_\beta\} = 0 \Rightarrow Q^2 = 0.$

$\Rightarrow Q$ is a BRST operator.

$\Psi_\mu = QA_\mu \Rightarrow Q\Psi_\mu = 0.$

A general $\mathcal{N}=2$ action has the form

$$S = \int d^{2n}\theta \underbrace{K(\Phi, \bar{\Phi})}_{\text{Kähler potential}} + \int d^n\theta\, W(\Phi) + \int d^n\bar{\theta}\, \bar{W}(\bar{\Phi})$$

superpotential, holom. in Φ.

<u>Main consequences of the topological twist</u>

1) $S^{SYM}_{top} = \{Q, \Psi\}$ action is Q-exact.

$+ 2\pi i \tau_0 \int tr(F \wedge F)$,

where $\tau_0 := \frac{\theta_0}{2\pi} + \frac{4\pi i}{g_0^2}$

This is equivalent to

$$S^{SYM}_{top} = Q\Psi + 2\pi i \tau_0 \int tr(F \wedge F)$$

<u>Useful identity</u>

$$\int_X tr\, F^\pm \wedge F^\pm = 2 \int_X tr(F \wedge * F) \pm tr(F \wedge F)$$

9) Energy-momentum tensor:

$$T_{\mu\nu} := \frac{1}{\sqrt{\det g}} \frac{\delta}{\delta g_{\mu\nu}} S^{SYM}_{top} = \{ Q, \Lambda_{\mu\nu} \}$$

for some $\Lambda_{\mu\nu}$

Hence, the path integral does not depend on the background metric.

Some comments on vdim Masd

Anomaly is proportional to Ind ($\not{D}_A$) =

= expression in Chern numbers of some spin bund

= $u(1)_R$ charge

= ghost number

= degree of diff. forms on Masd.

Anomaly is a symmetry which does not survive any quantization procedure

V. Pidstrygach

Cohomological descent

$W(\mathfrak{g}) = \Lambda^\bullet(\mathfrak{g}^\vee) \otimes S^\bullet(\mathfrak{g}^\vee)$

Super-Lie-alg. $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \langle d \rangle_1 = \tilde{\mathfrak{g}}$

$\iota_\xi \quad \mathcal{L}_\xi \quad d \qquad \mathcal{L}_\xi = [d, \iota_\xi]$

Ex 1 $G \circlearrowright M$ $\Omega^\bullet(M)$ is a representation of $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \langle d \rangle_1$

Ex 2 Basis of $\mathfrak{g}^\vee$: Θ^i deg $= 1$ $\quad\|\quad \mathcal{L}_a = \mathcal{L}_{K\Theta^a}$

z^i deg $= 2$ $\quad\|\quad \iota_a = \iota_{K\Theta^a}$

$d_K \Theta^i = z^i, \quad d_K z^i = 0$

$\mathcal{L}_a \Theta^b = -\sum_k c^b_{ak} \Theta^k \; ; \quad \mathcal{L}_a z^b = -\sum_k c^b_{ak} z^k$

$\iota_a \Theta^b = \delta_{ab} \qquad i_a z^b = \iota_a d\Theta^b = \mathcal{L}_a \Theta^b - d i_a \Theta^b = $
$= -\sum c^b_{ak} \Theta^k$

A alg, $\tilde{\mathfrak{g}} \to \mathrm{Der}(A)$

$A_{hor} = \{ \alpha \in A \mid i_a \alpha = 0 \ \forall a \}$

$A^G = \{ \alpha \in A \mid \mathcal{L}_a \alpha = 0 \ \forall a \}$

G acts freely on P, then $\Omega^\bullet(P)^G_{hor} \cong \Omega^\bullet(P/G)$

$$\boxed{\mu^b = z^b + \frac{1}{2}\sum c^b_{jk}\theta^j\theta^k} \quad (*)$$

deg = 2 element

Since $z^b = \mu^b - \frac{1}{2}\sum c^b_{jk}\theta^j\theta^k$,

$\{\mu^b, \theta^a\}$ can be taken as generators.

$$W(y) \cong \Lambda(\theta^1,..,\theta^n) \otimes S(\mu^1,..,\mu^n)$$

<u>Claim</u> μ^j are horizontal

$$i_a\mu^b = i_a z^b + \frac{1}{2}\sum_{j,k} c^b_{jk}\left(i_a\theta^j\cdot\theta^k - \theta^j i_a\theta^k\right)$$

$$= -\sum_k c^b_{ak}\theta^k + \frac{1}{2}\sum_{j,k}\left(\delta_{aj}\theta^k + \delta_{ak}\theta^j\right)$$

$$= -\sum_k c^b_{ak}\theta^k + \frac{1}{2}\sum c^b_{ak}\theta^k - \frac{1}{2}\sum \underset{-c^b_{aj}}{\underset{\shortparallel}{c^b_{ja}}}\theta^j = 0.$$

$$d_K\theta^a = \mu^a - \frac{1}{2}\sum c^a_{jk}\theta^j\theta^k \quad (**)$$

$$\mathcal{L}_a\mu^l = \mathcal{L}_a z^l + \frac{1}{2}\mathcal{L}_a\sum_{j,k} c^b_{jk}\theta^j\theta^k$$

$$\overset{\text{Jacobi}}{=} -\sum c^b_{ak}\mu^k$$

$$\boxed{d_K\mu^a = -\sum c^a_{jk}\theta^j\mu^b = \left(\sum \theta^b\mathcal{L}_b\right)\mu^a} \quad (***)$$

Since μ^a generate $y^\vee \subset S^\cdot(y^\vee) \Longrightarrow$ Bianchi identity

$\Rightarrow d_K \omega = \sum(\theta^b L_b)\,\omega$

"Supersymmetric change of variables" (*)

Often, one takes (**) & (***) as definitions

If this is the approach, one has to prove

$d_K^2 = 0$ and acyclicity

$$H^i(W(\mathfrak{g}), d_K) = \begin{cases} \mathbb{R}, & i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Mathai-Quillen iso Φ

$\tilde{\mathfrak{g}} \to \mathrm{Der}\, A$

$\tilde{\mathfrak{g}} \to \mathrm{Der}\, B$, A, B Algebras

Assume $\exists\, \hat{\theta}_b \in A$ s.t. $\iota_a \hat{\theta}^b = \delta_{ab}$

Derivation $\gamma = \sum \hat{\theta}^a \otimes \iota_a \in \mathrm{Der}(A \otimes B)$

If $n = \dim \mathfrak{g}$, then $\gamma^{n+1} = 0$

$\Phi = e^\gamma$ $\quad \mathrm{ad}_\gamma$, conj_Φ $\circlearrowleft$ $\mathrm{Der}(A \otimes B)$

$\in \mathrm{Aut}(A \otimes B)$

$\mathrm{ad}_\gamma(\delta) = [\gamma, \delta]$, $\delta \in \mathrm{Der}(A \otimes B)$

$\mathrm{conj}_\Phi(\delta) = \Phi\, \delta\, \Phi^{-1}$

$$\Phi(1 \otimes i_a + i_a \otimes 1)\Phi^{-1} = i_a \otimes 1$$

$$\Phi(d_A \otimes 1 + 1 \otimes d_B)\Phi^{-1} = \underbrace{d - \sum \mu^k \otimes i_k + \sum \theta^k \otimes L_k}_{\text{BRST}}$$

[Guillemin-Sternberg]

$$\Phi (A\otimes B)_{hor}\Phi^{-1} = A_{hor}\otimes B$$

Apply this to $A = W(\mathfrak{g})$ $B = \Omega^\bullet(M)$, $G \circlearrowright M$

$$\left(W(\mathfrak{g})\otimes\Omega^\bullet(M)\right)_{hor} = W(\mathfrak{g})_{hor}\otimes\Omega^\bullet(M)$$

$$= S^\bullet(\mathfrak{g})\otimes\Omega^\bullet(M)$$

~~Moreover~~ Furthermore

$$\left(W(\mathfrak{g})\otimes\Omega^\bullet(M)\right)^G = \left(S^\bullet(\mathfrak{g})\otimes\Omega^\bullet(M)\right)^G$$

$$d_K\otimes 1 + 1\otimes d_{dR} - \sum \mu^k\otimes i_k + \sum \theta^k\otimes \mathcal{L}_k\Big|_{(\ )^G_{hor}} =$$

$$= d_{dR} - \sum \mu^k\otimes \iota_k$$

$$=: \left(d_{dR} - \iota_{\mathfrak{g}}\right)$$

This is a Cartan model.

$\mathcal{A}_P$ con-ns on $P \to X^4$

$\mathcal{G} = \mathrm{Aut}(P)$

$$C^\infty(\mathcal{A}_P) \ni A^a_\mu(x)$$

$\Omega^1(\mathcal{A}_P\,;\,\Omega^1(P;\mathfrak{g})^G_{hor}) \ni \Psi$ tautological 1-form

$\underbrace{d_C}_{\text{Cartan differential}} A = \Psi$, where $A \in \Omega^0(\mathcal{A}_P\,;\,\Omega^1(P;\mathfrak{g})^G_{hor})$

$d_C \psi = -D_A \phi$ $\qquad \phi \in C^\infty(\mathrm{Lie}(\mathcal{G}), \Omega^0(P, \mathfrak{g}))^{\mathcal{G}}$

$d_c \phi = 0$ $\qquad \phi$ is the identity map.

$D_A \phi$ fundamental vector field of $\phi \in \mathrm{Lie}(\mathcal{G})$

Compare with the following:

$G \longrightarrow GL(\mathbb{R}^n)$

$\mathfrak{g} \longrightarrow \mathfrak{gl}_n(\mathbb{R})$

ξ_{ij}

$$K^\xi_x = \xi \cdot x$$

$$d_C^2(x^i) = d_C(dx^i) = (d - i_{K^\xi})dx^i$$

$$= -\sum \xi_{ij} x_j$$

<u>BRST</u> d_B, B for BRST

$$d_B A = \psi - D_A C$$

$$d_B \psi = [\psi, c] - D_A \phi$$

$$d_B c = \phi - \tfrac{1}{2}[c, c]$$

$$d_B \phi = -[c, \phi]$$

Pidstrygach 15.06 Göttingen

Cohomological descent II

$W^\cdot(\mathfrak{g})\otimes\Omega^\cdot(M)$ G-action on M

$\exists$ automorphism of $W(\mathfrak{g})\otimes\Omega^\cdot(M)$, which takes $\left(W(\mathfrak{g})\otimes\Omega^\cdot(M)\right)^G_{hor}$ to $\left(W(\mathfrak{g})\otimes\Omega^\cdot(M)\right)^G$

$$\left(W(\mathfrak{g})\otimes\Omega^\cdot(M)\right)^G = \underbrace{\left(S^\cdot(\mathfrak{g}^\vee)\otimes\Omega(M)\right)^G}_{\text{Cartan model}}$$

with the differential $d - i_{\mathfrak{g}}$

Free G-action on P (G princip. bundle)

$$\left(W(\mathfrak{g})\otimes\Omega^\cdot(P)\right)^G_{hor} \cong \big(\underbrace{W(\mathfrak{g})}_{\text{contractible}}\otimes\Omega(P)_{hor}\big)^G \longleftarrow \Omega(P)^G_{hor} = \Omega^\cdot(P/G)$$

induces an iso in cohomology

$W(\mathfrak{g}) = (\Lambda^\cdot\mathfrak{g}^\vee)\otimes S^\cdot(\mathfrak{g}^\vee)$, generators θ^a, μ^a

Contraction: $\delta\mu^a=\theta^a$, $\delta\theta^a=0$

Free action of the gauge gp $\mathcal{G}_P$ on $\mathcal{A}_P$

Cartan model

(A) $S^\cdot(\mathrm{Lie}(\mathcal{G}))\otimes\Omega^\cdot(\mathcal{A})$ $\quad d_c := d_{dR} - \iota_{\mathrm{Lie}(\mathcal{G})}$

$\sum p_i(\xi_i)\otimes\alpha_i$

$$\left(d-\iota_{\mathrm{Lie}(\mathcal{G})}\right)\sum_i p_i(\xi)\otimes\alpha_i = \sum p_i(\xi)\otimes d\alpha_i - \sum p_i(\xi)\otimes i_{K\xi}$$

$\xi\in\mathrm{Lie}(\mathcal{G}) = \Omega^0(\mathrm{ad}P)$

$P\to M$ The fund. v.f. on $\mathcal{A}_P$: $K_a^\xi = D_a\xi$

Changing the notation $\xi \rightsquigarrow \phi$

$$(d - i_{Lie(\phi)})(\alpha) = \sum_i p_i(\phi)\otimes d\alpha_i - \sum_i p_i(\phi)\otimes i_{K\phi}(\alpha_i)$$

$A \in C^\infty(\mathcal{A}; \Omega^1(P,\mathfrak{g})^G_{hor})$ | $d_C A = d_{dR} A = \Psi$ | $\deg \phi =$

$\Psi \in \Omega^1(\mathcal{A}; \Omega^1(P,\mathfrak{g})^G_{hor})$ | $d_C \Psi = -D_a \phi$ | $\deg A =$

$d_C \phi = 0$ | $\deg \Psi =$

$\phi \in \mathrm{Hom}(Lie(\mathfrak{g}), Lie(G))$ vector valued

identity map

G-invariant & closed elements of (A)

$\cancel{\mathcal{O}_2^{(0)}\phi(p)}$ $\mathcal{O}_2^{(0)}(p)$ $\pi(p)\in M$ $(p\in P\ \xi)$

$$\mathcal{O}_2^{(0)}(\pi(p)) := \frac{1}{8\pi^2}\mathrm{tr}(\phi^2(p))$$

$$\mathcal{O}_2^{(0)}(x) \in S^2(Lie(G)^\vee)$$

$$\mathcal{O}_2^{(1)}(\gamma) = \frac{1}{4\pi^2}\int_\gamma \mathrm{tr}(\phi\Psi) \qquad \mathcal{O}_2^{(1)} \in S^1(Lie(G)^\vee)\otimes\Omega^1(\mathcal{A}$$

$$\mathcal{O}_2^{(2)}(\Sigma) = \frac{1}{4\pi^2}\int_\Sigma \mathrm{tr}\left(\phi F_a - \frac{1}{2}\Psi\wedge\Psi\right), \quad \cancel{\mathcal{O}_2 \in S^1(Lie(G)^\vee)}$$

$$\mathcal{O}_2^{(2)} \in S^1(Lie(G)^\vee)\otimes\Omega^0(\mathcal{A}) \oplus \Omega^2(\mathcal{A})$$

All these elements are G-invariant

$$d_C\,\mathrm{tr}(\phi^2(p)) = 2\,\mathrm{tr}(\phi(p)\,d_C\phi(p)) = 2\,\mathrm{tr}(\phi \underbrace{d_C\phi}_{=0})(p) = 0$$

$$\cancel{2\,\mathrm{tr}(\phi D_A\phi)(p) = d^X \mathrm{tr}(\phi^2)(p)}$$

$$d_c \int_\gamma \mathrm{tr}(\phi\psi) = \int_\gamma \mathrm{tr}\left(d_c\phi\cdot\psi + \phi\, d_c\psi\right)$$

$$= \int_\gamma \mathrm{tr}\left(\phi\,(-D_a\phi)\right) = \int_\gamma d^M \mathrm{tr}(\phi^2) \overset{\text{Stokes}}{=\!=} 0$$

$$d_c \int_\Sigma \mathrm{tr}\left(\phi F_A - \tfrac{1}{2}\psi\wedge\psi\right) = \int_\Sigma \left(\phi\, d_c F_A - d_c\psi\wedge\psi\right)$$

$$= \int_\Sigma \mathrm{tr}\left(\phi\wedge D_A\psi + D_A\phi\wedge\psi\right) = -\int_\Sigma d^M \mathrm{tr}(\phi\wedge\psi).$$

Donaldson invariants

Motivation:

$$\mathcal{M}_k \subset \mathcal{B}_k^*$$

Can show: $[\mathcal{M}_k] \in H_d(\mathcal{B}_k^*)$ is well defined

$d = \text{vdim}\, \mathcal{M}_k = 8k - 3(b_2^+ - b_1 + 1)$

$[\mathcal{M}_k]$ can be thought ~~of~~ as an invariant, but this is not conv

If $\omega \in H^d(\mathcal{B}_k^*)$, then $\langle \omega, [\mathcal{M}_k] \rangle \in \mathbb{Z}$

Q: $H^\bullet(\mathcal{B}^*)$?

1. Slant product

Y, Z any top. spaces

$$/ : H^p(Z) \otimes H_q(Y \times Z) \to H_{q-p}(Y) \quad (*)$$

$$\cancel{\mathrm{Hom}(C_p(Z), \mathbb{Z}) \otimes C_q(Y \times Z)}.$$

Fact $\exists$ natural chain map

$$\xi : C_q(Y \times Z) \longrightarrow \sum_{s+t=q} C_s(Y) \otimes C_t(Z)$$

which is a chain homotopy equivalence

In particular, ξ induces an iso

$$\xi : H_q(Y \times Z) \longrightarrow H_q(C_\bullet(Y) \otimes C_\bullet(Z))$$

$$\mathrm{Hom}(C_p Z, \mathbb{Z}) \otimes C_q(Y \times Z) \qquad C_{q-p} Y \otimes C_p Z$$

$$\cap$$

$$\xrightarrow{\mathrm{id} \otimes \xi} \mathrm{Hom}(C_p Z, \mathbb{Z}) \otimes (C_\bullet Y \otimes C_\bullet Z)_q \longrightarrow \mathbb{Z}$$

$$\varphi \otimes (a \otimes b) \longmapsto \varphi(b)\, a$$

This is a chain map, hence we obtain $(*)$

<u>Properties</u>

<u>1° Naturality</u>

$$f : Y \to Y' \qquad u \in H^p(Z')$$

$$g : Z \to Z' \qquad v \in H_q(Y \times Z)$$

$$f_*\big(\underbrace{g^* u / v}_{H_{q-p}(Y)}\big) = u / (f \times g)_* v$$

2°. <u>Cap product</u>

$$\begin{tikzcd}[column sep=small] & Y \times Z \arrow[dl, "\pi_Y"'] \arrow[dr, "\pi_Z"] & \\ Y & & Z \end{tikzcd} \qquad \begin{array}{l} u \in H^p(Z) \\ v \in H_q(Y \times Z) \end{array}$$

$$u / v = (\pi_Y)_*(\pi_Z^* u \cap v)$$

<u>Ex</u> ~~[illegible]~~

Z cmpt oriented mf'd of dim $= d$; Y a p' mfld of dim $= n$

$$H^d(Z) \otimes H_q(Y \times Z) \longrightarrow H_{q-d}(Y)$$

$$\begin{array}{c} \wr\wr \\ \mathbb{Z} \end{array} \qquad H_q(Y \times Z) \longrightarrow H_{q-d}(Y)$$

$$\begin{array}{ccc} H_\bullet(Y\times Z) & \longrightarrow & H_{\bullet-1}(Y) \\ \downarrow PD & & \downarrow PD \\ H^{n+d-q}(Y\times Z) & \xrightarrow{\cong} & H^{n-q}(Y) \end{array}$$

integration along the fibers

de Rham - version:

$\omega \in \Omega^{k+d}(Y\times Z)$

$v_1,.., v_k \in \mathfrak{X}(Y)$

$$\int_Z i_{v_1}\dots i_{v_k}\omega = \Omega(v_1,..,v_k)$$

$$\Omega^{k+d}(Y\times Z) \longrightarrow \Omega^k(Y)$$
$$\omega \longmapsto \Omega$$

induces $\underbrace{H^{k+d}_{dR}(Y\times Z) \xrightarrow{\cong} H^k_{dR}(Y)}_{\text{integration along fibers}}$.

2. The universal bundle

$P \to X \ni x_0$ basept

$\mathcal{G}_0 = \{ g \in Aut(P) \mid g_{x_0} = id \}$ based gauge gp

$\{1\} \to \mathcal{G}_0 \longrightarrow \mathcal{G} \xrightarrow{ev_{x_0}} Aut(P_{x_0}) = G \to 0$ exact

$\mathcal{G}_0$ acts freely on $\mathcal{A}$

$\tilde{\mathcal{B}} = \mathcal{A}/\mathcal{G}_0$ mfld

DI 4

$G = \mathcal{G}/\mathcal{G}_0 \curvearrowright \tilde{\mathcal{B}} \longrightarrow \mathcal{B}$

$\cancel{\tilde{\mathcal{B}}/G}$

$$\begin{array}{ccc} \mathcal{G} \curvearrowright \ \mathbb{P} := \pi_2^* P & \longrightarrow & P \\ \downarrow & & \downarrow \\ \mathcal{G} \curvearrowright \mathcal{A} \times X & \xrightarrow{\pi_2} & X \end{array}$$

$\tilde{\mathbb{P}} = \mathbb{P}/\mathcal{G}_0 \longrightarrow \mathcal{A} \times X/\mathcal{G}_0 =: \tilde{\mathcal{B}} \times X$ universal bundle

$$/ : H^d(\tilde{\mathcal{B}} \times X) \times H_\ell(X) \longrightarrow H^{d-\ell}(\tilde{\mathcal{B}})$$

$$\underset{c_{d/2}(\tilde{\mathbb{P}})}{\cup}$$

$$\tilde{\mu}_d : H_\ell(X) \to H^{d-\ell}(\tilde{\mathcal{B}})$$
$$\alpha \longmapsto c_{d/2}(\tilde{\mathbb{P}})/\alpha$$

<u>Prop</u>

$$\left.\begin{array}{l} P \to X \quad SU(2) \text{ bundle} \\ X \quad 1\text{-connected} \\ \Sigma_1, \dots, \Sigma_\ell \text{ basis of } H_2(X;\mathbb{Q}) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} H^*(\tilde{\mathcal{B}};\mathbb{Q}) \text{ is generated} \\ \text{by } \tilde{\mu}_2(\Sigma_1), \dots, \tilde{\mu}_2(\Sigma_\ell) \\ \text{as a ring, i.e.} \\ H^*(\tilde{\mathcal{B}};\mathbb{Q}) \cong S^*(H_2(X;\mathbb{Q})) \\ H^{2k}(\tilde{\mathcal{B}};\mathbb{Q}) \cong S^k(H_2(X;\mathbb{Q})) \end{array}\right.$$

$$\tilde{\mu}_2(\Sigma_j) = c_2(\tilde{\mathbb{P}})/\Sigma_j \in H^2(\tilde{\mathcal{B}};\mathbb{Q})$$

<u>∴ wl</u>: $H^*(\mathcal{B}^*;\mathbb{Q})$?

DIS

$\tilde{\mathcal{B}} \supset$ & the action of $G/\{\pm 1\}$
is free only on $\hat{\mathcal{B}}^*$

$$\begin{array}{ccc} \mathbb{P}^{ad} & \leftarrow - & \tilde{\mathbb{P}} \\ \vdots & & \downarrow \\ \mathcal{B}^* & \leftarrow - & \hat{\mathcal{B}}^* \end{array}$$

Sur. ... bu ...

... above.

$$\mu : H_2(X;\mathbb{Q}) \longrightarrow H^2(\mathcal{B}^*;\mathbb{Q})$$
$$\Sigma \longmapsto -\tfrac{1}{4}p_1(\mathbb{P}^{ad})/\Sigma$$

$SO(3) \hookrightarrow \tilde{\mathcal{B}}^*$

$\downarrow$

$\mathcal{B}^*$

$$\nu := -\tfrac{1}{4}p_1(\tilde{\mathcal{B}}^* \to \mathcal{B}^*) \in H^4(\mathcal{B}^*;\mathbb{Q})$$

Prop

$$H^\bullet(\mathcal{B}^*;\mathbb{Q}) \cong \mathbb{Q}[\nu, \mu(\Sigma_1), \dots, \mu(\Sigma_b)] \text{ as rings.}$$

Rem.
$$/ : \underset{\substack{\psi \\ -\frac{1}{4}p_1(\mathbb{P}^{ad})}}{H^4(\mathcal{B}^* \times X)} \times H_0(X) \longrightarrow H^4(\mathcal{B}^*)$$

$$\mu_4 : H_0(X) \longrightarrow H^4(\mathcal{B}^*)$$
$$\alpha \longmapsto -\tfrac{1}{4}p_1(\mathbb{P}^{ad})/\alpha$$

$H_0(X) = \mathbb{Q}[x_0]$ & $\mu_4[x_0] = -\tfrac{1}{4}p_1(\mathbb{P}^{ad})/[x_0] = \nu$

3. The Donaldson invariant

DI 6

Neo smartpen

Assume first $\operatorname{vdim} \mathcal{M}_k = 8k - 3(b^+ - b_1 + 1) = 0$

Assume

$\pi_1(X) = \{1\} \Rightarrow b_1 = 0$

$b_2^+ \geq 2$

Want:

(i) $\mathcal{M}_k$ contains no reducible solutions

(ii) $\mathcal{M}_k$ is a finite set of pts cut out transversally

(iii) Can attach signs ± 1 to each pt in $\mathcal{M}_k$

(i): holds for a generic metric on X, since $b_2^+ \geq 2$

(ii): $\overline{\mathcal{M}}_k = \mathcal{M}_k \cup (\mathcal{M}_{k-1} \cup X) \cup \dots$

$$\operatorname{vdim} \mathcal{M}_{k-1} = 8(k-1) - 3(b^+ + 1) = \underbrace{8k - 3(b^+ + 1)}_{= 0} - 8 < 0$$

For generic g, $\mathcal{M}_{k-1} = \emptyset$

$\Rightarrow \overline{\mathcal{M}}_k = \mathcal{M}_k$ cmpt

g generic $\Rightarrow \mathcal{M}_k$ is cut out transversally

(iii) Roughly, $\varepsilon([A]) := \operatorname{sign} \underline{\det(d_A^{+} + d_A^{*})}$ is not well-defined

However, if A_0, A_1 are two solutions, we can define $\varepsilon(A_0, A_1) \in \{\pm 1\}$ s.t.

"$\operatorname{sign} \det(d_{A_1}^+ + d_{A_1}^*) = \varepsilon(A_0, A_1) \operatorname{sign} \det(d_{A_0}^+ + d_{A_0}^*)$"

so [illegible] A is [illegible] [illegible] an overall sign.

AB.

$$\boxed{q = \sum_{[A] \in \mathcal{M}_k} \epsilon(A) \in \mathbb{Z}}$$

The Donaldson invariant

More generally, ~~if vdim $\mathcal{M}_k > 0$, define~~ assume $2d = \text{vdim}\, \mathcal{M}_k > 0$, $d \in \mathbb{N}$

"Define"

$q_k : S^k(H_2(X;\mathbb{Z})) \longrightarrow \mathbb{Z}$

$$q_k(\Sigma_1, \dots, \Sigma_d) = \langle \mu(\Sigma_1) \cup \dots \cup \mu(\Sigma_d), [\mathcal{M}_k] \rangle$$

Main problem: $\mathcal{M}_k$ is noncompact.

Main property: Naturality

If $f : X \to Y$ orientation preserving diffeo,

$$q_k(f_* \Sigma_1, \dots, f_* \Sigma_d) = q_k(\Sigma_1, \dots, \Sigma_d)$$

Pidstrygach. 15.06. Göttingen

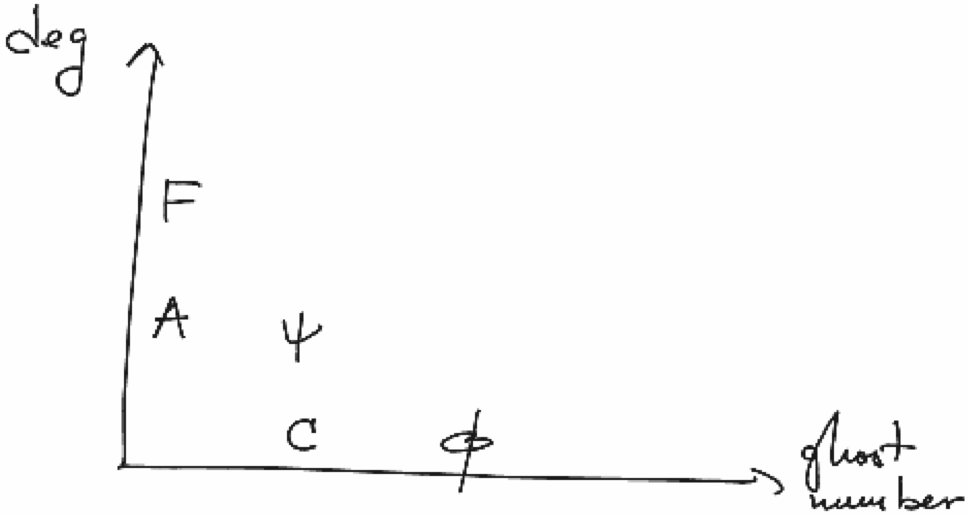
Donaldson polynomials & Observables

BRST A, ψ, ϕ as in the Cartan model

$c \in \mathrm{Lie}(\mathcal{G}) = \Omega^0(P; \mathfrak{g})^G$, $\deg c = 1$

$$\left.\begin{aligned} d_B A &= \psi - D_A c \\ d_B \psi &= [\psi, c] - D_A \phi \\ d_B c &= \phi - \tfrac{1}{2}[c, c] \\ d_B \phi &= -[c, \phi] \end{aligned}\right\} (*)$$

$$\begin{tikzcd} & \mathcal{E}^+ = \Omega^2_+(P; \mathfrak{g}) \arrow[d] \\ & \mathcal{A}^* \arrow[u, bend left, dashed, "\mathcal{F}"] \arrow[d] \\ \mathcal{M} \subset & \mathcal{A}^*/\mathcal{G} = \mathcal{B}^* \end{tikzcd}$$



Consider $\mathbb{A} = A + c$ as a new connection

$W(\mathfrak{g}^\vee) \longrightarrow \Omega^\bullet(P)$

$\theta^a \longmapsto$ comp. of a conn

$\mu^a \longmapsto$ comp. of curvature

$$F_{\mathbb{A}} = F_A + \psi + \varphi$$

$$\left\{\begin{array}{ll} (d+d_B)\,\mathbb{A} = \mathbb{F} - \frac{1}{2}[\mathbb{A},\mathbb{A}] & \text{defn of curv} \\ (d+d_B)\,\mathbb{F} = [\mathbb{A},\mathbb{F}] & \text{Bianchi identity} \end{array}\right. \Leftrightarrow (*)$$

$$\begin{array}{c} \mathbb{P} \\ \downarrow \\ \mathcal{B}\times X \end{array} \qquad c_2(\mathbb{P}) = \nu\otimes 1 + \sum_{\Sigma_i\in H_2(X;\mathbb{Z})} \mu_{\Sigma_i}\otimes[\Sigma_i]^{PD} + 1\otimes c_2(P$$

vol

There is a conn $A+C$ on the principal bundle $\mathbb{P}$, which is $A+C \in \Omega^1(\mathcal{A}\times P, Lie(\mathcal{G})\oplus \mathfrak{g}$

The curvature is $\phi + \psi + F_A$

<u>Andreas Ikkert</u> <u>The Donaldson-Witten partition function</u>.

QFT-heuristics in $N=2$ SUSY-YM.

<u>Def</u> Corr. function

$$\langle \mathcal{F}\rangle = \langle \mathcal{F}\rangle_T = \int [dA\ldots]\, e^{S_T}\,\mathcal{F}$$

for $$S_T = -\frac{1}{g_0^2}\int_X (-\operatorname{tr} F^+\wedge F^+ + \ldots)$$

$$= -\frac{1}{g_0^2}\{Q,\Psi\}$$

<u>Lemma</u> a) $\langle\{Q,\mathcal{O}\}\rangle = 0$ for every field expressio

b) $\delta_{g_0^2}\langle\mathcal{O}\rangle = \langle\delta_{g_0^2}\mathcal{O}\rangle$

c) $\alpha \{Q, \mathcal{O}\} \Rightarrow \delta_{g_{\mu\nu}} \langle \mathcal{O} \rangle = \langle \delta_{g_{\mu\nu}} \mathcal{O} \rangle$

When is $\langle \mathcal{O} \rangle$ invariant (under diffeomorphisms)?

$$\left.\begin{array}{l} \text{(i) } \{Q, \mathcal{O}\} = 0 \\ \text{(ii) } \mathcal{O} \neq \mathcal{O}(g_{\mu\nu}) \end{array}\right\} \Rightarrow \delta_{g_{\mu\nu}} \langle \mathcal{O} \rangle = 0.$$

(iii) If $\langle \mathcal{O} \rangle \neq 0 \Rightarrow \mathcal{O} \neq \{Q, e\}$ for some e

This is satisfied by $\mathcal{O}^{(0)}(p) = \mathrm{Tr}(\phi^2(p))$

<u>Lemma</u> $x_1 \longrightarrow x_2$ location change is Q-exact

$$\frac{\partial}{\partial x^\mu} \mathcal{O}^{(0)}(x) = 2i \{Q, \mathrm{Tr}\, \phi(x) \psi_\mu(x)\}$$

$$\mathcal{O}^{(0)}(p_1) - \mathcal{O}^{(0)}(p_2) = \int_\gamma \underbrace{\frac{\partial \mathcal{O}^{(0)}}{\partial x^\mu} dx^\mu}_{d\mathcal{O}}$$

$$= i \{Q, \underbrace{\int \mathrm{Tr}(\phi \psi_\mu) dx^\mu}_{\mathcal{O}^{(1)}}\}$$

<u>Consider</u> $d\mathcal{O}^{(i)} = i\{Q, \mathcal{O}^{(i+1)}\}$ $i = 0, \dots, 3$

$$0 = i\{Q, \mathcal{O}^{(4)}\}, \quad d\mathcal{O}^{(4)} = 0$$

Explicitly,

$$\mathcal{O}^{(1)} = \mathrm{Tr}(\phi \wedge \psi)$$

$$\mathcal{O}^{(2)} = \mathrm{Tr}(\psi \wedge \psi + 2i \phi \wedge F)$$

$$\mathcal{O}^{(3)} = 2i\, \mathrm{Tr}(\psi \wedge F), \quad \mathcal{O}^{(4)} = -\mathrm{Tr}(F \wedge F)$$

For $\Sigma_i \in H_i(X)$ set $\mathcal{O}(\Sigma_i) := \int_{\Sigma_i} \mathcal{O}^{(i)}$

Fact a) $\mathcal{O}(\Sigma_i)$ are Q-closed

b) $gh \# \mathcal{O}^{(j)} = 4 - j$

c) $\delta_{g_{\mu\nu}} \mathcal{O}(\Sigma_i) = 0$

Thm For $k_1, .., k_r \in \{0, .., 4\}$ and $\Sigma_{k_j} \in H_{k_j}(X)$

$$\delta_{g_{\mu\nu}} \langle \mathcal{O}(\Sigma_{k_1}) \cdots \mathcal{O}(\Sigma_{k_r}) \rangle = 0 .$$

In the weak coupling limit (g_0 small) the path integral is dominated by classical minima ($\mathrm{tr}\, F^+ \wedge F^+$ pos. definite) $\Leftrightarrow F^+ = 0$.

- path integral can be evaluated in $\mathcal{M}_{ASD}(P)$.

Lemma $n = \dim \mathcal{M}_{ASD}(P)$, in the wcl we have

a) $n = \#$ of 0 modes of ψ $- \underbrace{\# \text{ of } 0 \text{ modes } (\eta, \chi)}_{= 0, \; G = SU(2)}$

b) $\langle \mathcal{O} \rangle \neq 0 \Leftrightarrow \langle \mathcal{O} \rangle = \langle \underbrace{\Theta_{i_1 \dots i_n}}_{\substack{n \text{ form} \\ \text{on the space} \\ \text{of connections}}} \underbrace{\psi^{i_1} \dots \psi^{i_n}}_{0 \text{ modes}} \rangle$

$\Rightarrow [DA \dots] \to dA_1 \dots dA_n d\psi_1 \dots d\psi_n$

Thm For $k_1, .., k_r \in \{0, .., 4\}$ and $\Sigma_{k_j} \in H_{k_j}(X)$ (so that $n = \sum (4 - k_j)$) there exists a k_j-form Θ_{Σ_j} on $\mathcal{M}$ s.t.

$$\langle \mathcal{O}(\Sigma_{k_1}) \ldots \mathcal{O}(\Sigma_{k_n}) \rangle = \int_{\mathcal{M}} \Theta_{\Sigma_1} \wedge \ldots \wedge \Theta_{\Sigma_k}$$

Proof Sketch

Problem: $\mathcal{O}^{(0)}, \mathcal{O}^{(1)}, \mathcal{O}^{(2)}$ contain ϕ! ~~...~~ Eliminate ϕ by $\langle \phi \rangle$, which are expressable in terms of other fields

$$\Theta_{\Sigma_0} = \mathrm{Tr} \langle \phi(\Sigma_0) \rangle$$

$$\Theta_{\Sigma_1} = \alpha \int_{\Sigma} \mathrm{Tr}(\langle \phi \rangle \wedge \psi)$$

Witten shows

$$\langle \phi^a(x) \rangle = -i \int_X dy \, \underbrace{G^{ab}(x,y)}_{\text{Green's function}} [\psi_\alpha(y), \psi^\alpha(x)]$$

$$\Theta_{\Sigma_2} = \int_{\Sigma_2} \mathrm{Tr}(\psi \wedge \psi + 2i \langle \phi \rangle \wedge F)$$

$$\Theta_{\Sigma_3} = \alpha i \int_{\Sigma_3} \mathrm{Tr}(\psi \wedge F)$$

$$\Theta_{\Sigma_4} = -\zeta \int_X \mathrm{Tr}(F \wedge F), \quad \zeta \in \mathbb{R}$$

---

2. Donaldson-Witten partition function

$$\mathcal{O} := \mathcal{O}(\Sigma_0), \quad \mathcal{O}(\Sigma) := \mathcal{O}(\Sigma_2)$$

Def For $p \in H_0(X)$ & $\Sigma \in H_2(X)$

$$Z_{DW}(p, \Sigma) := \sum_{\substack{m \\ \text{instanton factor}}} \langle e^{p\mathcal{O} + \mathcal{O}(\Sigma)} \rangle = \sum_{l,2} \frac{p^l}{l!\,2!} \sum_m \langle \mathcal{O}^l \mathcal{O}(\Sigma)^2 \rangle,$$

<u>Note</u> For $\mathcal{O}^l \mathcal{O}(\Sigma)^r$ only the instanton sector with $\dim \mathcal{M}_m = 4l + 2r$ contributes <u>one</u> term

<u>Wittens claim</u>: $\mu_d(\underset{\Sigma_0}{\overset{\shortparallel}{p}}) \longleftrightarrow \mathcal{O}_p$ and

$\mu_d(\Sigma) \longleftrightarrow \mathcal{O}_\Sigma$ s.t.

$$Z_{DW}(p, s) := \sum_{l, r \geq 0} \frac{p^l}{l! r!} \underbrace{\langle \mathcal{O}^l \mathcal{O}(\Sigma)^r \rangle}_{P_W(\mathcal{O}^l \mathcal{O}^r)}$$

$$= \frac{1}{2} \Lambda^{-3/4(\chi + \sigma)} \sum_{l, r \geq 0} \frac{p^l s^r}{l! r!} \Lambda^{2l + |r|} P_D(p^l \Sigma^r)$$

$p \in H_0(X)$, $p \in H^0(X)$ dual element

$\mathcal{O}(\Sigma) = \sum_r s^\mu \mathcal{O}(\Sigma_r)$, $s^\mu \in H^2(X)$

$s^r = \prod_\mu (s^\mu)^{r_\mu}$ with $r = \sum r_\mu$

Pascual. 26.06.18

Monopole eqns

Gauge theory

G Lie gp (structure gp), $\mathfrak{g} = \mathrm{Lie}(G)$

$P \to M$ principal G-bundle

$\mathcal{G} = \mathrm{Aut}(P)$ gauge gp

$\mathcal{A} = \mathrm{Conn}(P)$

Cartan model $A \in C^\infty(\mathcal{A}, \Omega^1(P;\mathfrak{g})^G_{hor})$

$\Psi \in \Omega^1(\mathcal{A}, \Omega^1(P;\mathfrak{g})^G_{hor})$

A gauge field, Ψ superpartner (fermion field)

$\phi \in \mathrm{Hom}(\mathrm{Lie}(\mathcal{G}), \mathrm{Lie}(\mathcal{G}))$ scalar field

D Auxiliary field.

Today $D \sim F^+ = \mu(M)$?

Math: consider another rep. of G

Phys: Add another content

We can extend the $N=1$ theory to $N=2$

Consider repr. of $SU(2)_- \times SU(2)_+ \times SU(2)_R \times U(1)$,

Phys	Math
Vect: A_μ	$(2,2,1)^0$
Spinor $(\bar\Psi^A_{\dot\alpha}, \psi^A_\alpha)$	$(1,2,2)^{-1} \oplus (2,1,2)^1$
scalar $(\phi, \bar\phi)$	$(1,1,1)^2 \oplus (1,1,1)^{-2}$
D	

Vector multiplet (VM)

Partition f-n $\quad Z = \int [d\psi..] \, e^S$

Want to compute after the topological twist:

$$SU(2)_+ \times SU(2)_- \times U(1)_R$$

$$\downarrow$$

$$SU(2)_+ \times SU(2)_- \times SU(2)_R \times U(1)_R$$

$$(A, B, z) \mapsto (A, B, A, z)$$

After the top. twist spinors become vectors

(some) scalars spinors

Hypermultiplets

Take a cx rep. R of G

$R^\vee = R^* = \mathrm{Hom}(R, \mathbb{C})$

Define $W = R \oplus R^\vee$. On this we define a cx. str. J by the following rule

$\varphi \in R^\vee$, $\varphi = \langle \cdot, r_\varphi \rangle \qquad J(r, \varphi) = (r_\varphi, -\langle \cdot, r \rangle)$

With $I = i$ from R and $K = IJ$
we gain a quaternionic structure on W.
Instead of $R \oplus R^\vee$ take $(r, r_\varphi) =: M \in R \oplus \bar{R}$.
M is transforming in R (scalar)

$$\begin{array}{l} P \leftarrow G \\ \downarrow \\ X \end{array} \qquad \text{Define } \mathcal{R} := P \times_G R$$

Now call $S_\pm$ the $SU(2)_\pm$ spin bundles
Together and after the twist we get $\hat{M} = (M_1, M_2)$

$$(M_1, M_2) \in \Gamma(S^+ \otimes \mathcal{R}) \oplus \Gamma(S^+ \otimes \mathcal{R})$$

- If X is not spin, we need $w_2(\mathcal{R}) = w_2(X)$
- Even if there is no $\mathcal{R}$ as bundle, we can construct $S^+ \otimes \mathcal{R}$.

<u>Moment map</u> (after twist also auxiliary field)

<u>Recall</u>: $\forall \xi \in \mathfrak{g}$ $\quad K_\xi \in \mathfrak{X}(X)$ $\quad , K_\xi(x) = \frac{d}{dt}\Big|_{t=0} (\exp(t\xi) x)$

For I, J, K we define 2-forms

$$\omega_I(\cdot, \cdot) = \langle I \cdot, \cdot \rangle$$
$$\omega_J(\cdot, \cdot) = \langle J \cdot, \cdot \rangle$$
$$\omega_K(\cdot, \cdot) = \langle K \cdot, \cdot \rangle$$

Since $i_{K_\xi} \omega_{I,J,K}$ is exact, we have μ_ξ^i s.t. $d\mu_\xi^i = i_{K_\xi} \omega$
Together these define a moment map $\mu : W \to \mathfrak{g}^\vee \otimes \mathbb{R}^3$

Now we can write down the monopole equations (Q fixed pts)

$$\begin{cases} F^+ = \mu(M) \\ \not{D} M = 0 \end{cases} \qquad \not{D} : \Gamma(S^+ \otimes \mathcal{R}) \to \Gamma(S^- \otimes \mathcal{R})$$

<u>Example</u> $G = U(1)$, then $\mathcal{R} = \mathcal{L}$ line bundle and the monopole equs are classical Seiberg-Witten equs.

$$\begin{cases} F^+_{\dot\alpha\dot\beta} = \overline{M}_{(\dot\alpha} M_{\dot\beta)} \\ \not{D}_{\alpha\dot\beta} M^{\dot\beta} = 0 \end{cases}$$

$$\varepsilon_{\dot\alpha\dot\beta} M^{\dot\beta} = M_{\dot\alpha}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} -M \\ M \end{pmatrix}$$

<u>Recall</u>: For $S \sim \underbrace{\frac{1}{e^2}}_{\text{coupling constant}} \int$ = action

e is dependent on additional data Λ, "energy scale". The evolution of e is given by β-function: $\Lambda \frac{d}{d\Lambda} e = \beta$

From physics we know β is non-positive (asymptotically free theory)

In our case

$$\beta \sim -2h^\vee + c_2(R)$$

For generators X_i of $\mathfrak{g}$ $c_2(R) = \sum_i X_i X_i$

So we ~~the~~ should have $\boxed{2h^\vee < c_2(R)}$

<u>Outlook</u> We can compute Donaldson's invariants with the new theory.
For this we need to map these UV ideas to IR (low Λ or norms of the fields)

infrared (IR) limit $\rightsquigarrow$ weak cx integrable systems

Elliptic curves

∇ ell. curve over $\mathbb{C}$ is Riemann surface $= \mathbb{C}/\Lambda$,

Λ is a lattice $\Lambda = \mathbb{Z}\langle a, b\rangle$, a, b indep. over $\mathbb{R}$

s.t. $\mathbb{R}\cdot a + \mathbb{R}\cdot b = \mathbb{C}$

∘) $\wp(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \frac{1}{(z-w)^2} - \frac{1}{w^2}$

propert.: meromorphic on $\mathbb{C}$, poles at $w \in \Lambda$, ord 2 with index 0

∘) $\wp(z+w) = \wp(z) \quad \forall w \in \Lambda$

∘) $\exists g_2 = g_2(\Lambda), g_3 = g_3(\Lambda)$ const. s.t.

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

$$E = \{[x:y:1] \in \mathbb{CP}^2 \mid y^2 = 4x^3 - g_2x - g_3\}$$

<u>Thm</u> The map $\mathbb{C} \longrightarrow E$

$$z \longmapsto [\wp(z); \wp'(z); 1]$$

is biholomorphic and induces $\mathbb{C}/\Lambda \cong E$

Conversely, $\forall g_2, g_3 \in \mathbb{C}$: $\Delta = -16(-4g_2^3 + 27g_3^2) \neq 0$

(i.e. E is smooth) $\exists \Lambda \subset \mathbb{C}$ s.t. $E \cong \mathbb{C}/\Lambda$

genus $E = 1 \Rightarrow \dim H^0(E;\Omega^1) = 1$
$\omega \neq 0$ canonical bdl

lin map $\psi: H_1(E;\mathbb{Z}) \to \mathbb{C}$
$\gamma \longmapsto \int_\gamma \omega$

One can show $H_1(E;\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}$,
then $\operatorname{Im}\psi = \Lambda$ is a lattice in $\mathbb{C}$

$\Psi : E \longrightarrow \mathbb{C}/\Lambda$, $p \longmapsto \int_{p_0}^{p} \omega$

Rem •) ω can be given explicitly: $\omega = \frac{dx}{y}$
•) $\pi_1(\mathbb{C}) = 0 \Rightarrow \pi_1(E) = \Lambda = H_1(E;\mathbb{Z})$
$H_1(E;\mathbb{Z}) \times H_1(E;\mathbb{Z}) \longrightarrow \mathbb{Z}$
$(\alpha,\beta) \longmapsto$ ~~Signature of~~ sum of oriented intersection pairs

D Λ & Λ' are equivalent $:\Leftrightarrow \exists b \in \mathbb{C}^\times : \Lambda = b\cdot\Lambda'$

$\Lambda = \langle a, b\rangle = a^{-1}\langle 1, b/a\rangle =: \Lambda_{b/a}$
it is oriented $\Leftrightarrow b/a \in \mathbb{H} = \{\operatorname{Im} z > 0\}$

$\Lambda_\tau \cong \Lambda_{\tau'} \Leftrightarrow \exists A \in SL_2\mathbb{Z} : \tau = A\tau' = \frac{a\tau'+b}{c\tau'+d}$

Prop
$\mathbb{H}/SL_2\mathbb{Z}$ classifies elliptic curves over $\mathbb{C}$
$\cong$
$\mathbb{C}$

Rem We could define ell. curves by
$$y^2 = P_3(x), \deg P_3 = 3$$
We could choose other normal forms for P_3, e.g.
$$P_3(x) = x(x-1)(x-\lambda)$$

Special Kähler str.

(M, I) cx mf

D sp. Kähler str. is ~~Special~~ $(M, g, \omega, I, \nabla)$ s.t.

1) (g, ω, I) Kähler str.

2) ∇ conn (not necessarily ∇^{LC}) $\nabla\omega = 0$, flat, torsion free

3) $(\nabla_X I)Y = (\nabla_Y I)X$

D $(x_1, \dots, x_n, y_1, \dots, y_n)$ flat Darboux system

1) $\omega = dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n$

2) $\nabla dx_j = \nabla dy_j = 0$

Prop $\forall j$ $\exists$ holom. fct z_j on M & w_j on M

$\mathrm{Re}\, z_j = x_j$ $\qquad$ $\mathrm{Re}\, w_j = -y_j$

$(z_1, \dots, z_n)$ holo. coord & $(w_1, \dots, w_n)$ sp. holo coord
Sp. holo. coord

$(z), (w)$ conjugate sp. hol. coord., $\tau_{ij} = \dfrac{\partial z_i}{\partial w_j}$

$\tau_{ij} = \tau_{ji} \Rightarrow \exists F(z_1, \dots, z_n)$ s.t. $w_j = \dfrac{\partial F}{\partial z_j}$
holo. prepotential

$$\omega = \frac{i}{2} \sum \mathrm{Im}\, \tau_{jk}\, dz_j \wedge d\bar{z}_k,$$
$(\mathrm{Im}\, \tau_{jk})$ is pos. def.

D alg. integrable syst. :

$\pi : Z^{2n} \to M$ holo, Z, M cx mfld

a) Z is a cx mfd, $\eta \in \Omega^{2,0}(Z)$ cx sympl.
$(\Rightarrow \quad \eta^{\wedge n} \neq 0)$

b) $Z_m = \pi^{-1}(m)$ - cx Lagrangian submf
cpt cx tori

c) $\exists$ a family $[\rho_m]_{m \in M} \in H^{1,1}(Z_m) \cap H^2(Z_m; \mathbb{Z})$
positive polarisation of Z_m

Slogan: "alg. intg. systems" = "sp. kähler str. on M"

Thm the base of any alg. intg. system carries
a sp. kähler str.

$n = 1$ (is enough for us) Z cx surface
M Riemann surface

$p \in Z$, $V_p = T_p Z_{\pi(p)}$ vertical space
$= \ker D_p \pi$

η sympl. form induces
$V_p \xrightarrow{\cong} T^*_m M$, $\iota : T^*_m M \to V_p$
$v \longmapsto \eta_p(v, \cdot)$

$f : M \to \mathbb{C}$, $\eta(\iota(df), \xi) = \pi^* df(\xi)$

$\exp : T^*_m M \to Z_m$
$\searrow \quad \nearrow$
V_p

$\Lambda_m = \text{Stab}_p \subset T^*_m M$, $p \in \pi^{-1}(m)$ fixed

Fact: $\forall m \in M$, Λ_m is a lattice in $T^*_m M$.

$\Lambda \subset T^*M$, st $\Lambda_m = \Lambda \cap T^*_m M$

Λ is a Lagrangian submf wrt the canonical sympl. str. on T^*M.

$\forall m \in M$ choose U_m nbhd. $n = 1$

$(\alpha_1, \dots, \alpha_{2n})$ s.t. $\alpha_j(m') \in \Lambda_{m'}$
1-forms
local triv of $T^\vee M$ on U_m

Defin. ∇ by $\nabla \alpha_j = 0$ – flat on $T^\vee M$

as $\Lambda \subset T^\vee M$ is Lagrangian $\Rightarrow$ ∇ torsion-free

$\mathrm{Hol}_m \nabla \subset$ ~~Sp~~ $\mathrm{Aut}\, \Lambda_m$ ($\cong Sl_2\mathbb{Z}$ if $n = 1$)

(c) given $[\rho_m]$ s.t. $[\rho_m]$ is generating $H^2(Z_m; \mathbb{Z})$

Think of ρ as a 2-form on Z s.t.
$\rho|_V$ is an invariant volume form

$\tilde{T}M = V$ yields a sympl. str on $T_m M$

$T_m M$:
Skew-sym. integer valued bilin map $\Lambda_m \times \Lambda_m \to \mathbb{Z}$
$\langle \cdot, \cdot \rangle$

<u>Prop</u> $\exists$ Darboux coord. on Λ_m, i.e.
$\gamma_1, \dots, \gamma_n, \delta_1 \dots \delta_n$ s.t. $\langle \gamma_i, \gamma_j \rangle = \langle \delta_i, \delta_j \rangle = \cancel{\delta_{ij}}\, 0$
$\langle \gamma_i, \delta_j \rangle = \delta_{ij}$

$\Lambda \subset T^\vee M$ Lagrangian $\Rightarrow$ $\gamma_j = dx_j$, $\delta_j = dy_j$

Define $\omega = dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n$

<u>Thm 1</u> $(g = \omega(\cdot, I \cdot), I, \omega, \nabla)$ is special Kähler

<u>Thm 2</u> For any sp.K. str. on M $\exists$ alg. integr. syst.
which recovers the sp.K. from Thm 1
provided $\mathrm{Hol}\, \nabla \subset Sl_2\mathbb{Z}$

How do we get sp. holo. coord?

Put: $dz_i := \int_{\gamma_i} \eta$, $dw_j := -\int_{\delta_j} \eta$

The pentagon example: $M = \mathbb{C} \setminus \{\pm 2\Lambda^3\}$, $\Lambda \in \mathbb{C}$
may assume $\Lambda = 1$

$\Sigma_u = \{ y^2 = z^3 - 3\Lambda^2 z + u \}$ smooth on M

$\lambda = y\,dz$, $Z = \{ (y,z,u) | \dots \} \subseteq \mathbb{C}^2 \times \mathbb{C}$

$\Lambda_u = H_1(\Sigma_u; \mathbb{Z})$ lattice in $\mathbb{C}$

Choose γ_e, γ_m, $\langle \gamma_e, \gamma_m \rangle = \pm 1$

Notice: $z_e = \int_{\gamma_e} \lambda$ is sp. holo. coord.

Pidstrygach 26.06

Ghosts

Cartan model of equiv. cohomology $G \times V \to V$

$(S(\mathfrak{g}^\vee) \otimes \Omega^\cdot(V))^G$ cohomology of this are equiv. cohomol. of V.

$$(d - \iota_{\mathfrak{g}}) \sum p_i(\xi) \otimes \alpha_i = \sum \left(p_i(\xi) \otimes d\alpha_i - p_i(\xi) \otimes \iota_{K^\xi}\right)$$

Consider the case when V is a vector space equipped with a G-invariant metric

$\dim V = 2m$

$\rho : G \to \mathrm{Aut}(V)$

$\dim G = d$

$\hat{\rho} : \mathfrak{g} \to \mathrm{End}(V)$

G cmpt Lie gp.

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_{2m} \end{pmatrix}$ $\quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_{2m} \end{pmatrix}$

~~$U := \pi^{-m} e^{-|x|^2} \int \left(e^{x^t \hat{\rho}(\xi) y} + i dx^t y\right.$~~

$$U := \pi^{-m} e^{-|x|^2} \int \exp\left(\tfrac{1}{4} y^t \hat{\rho}(\xi) y + i\, dx^t y\right) \mathcal{D} y$$

U is the Mathai-Quillen element

$P \to X$ a conn

$a \in \Omega^1(P; \mathfrak{g})^G, \quad a(K^\xi) = \xi$

$$F_a = da + \tfrac{1}{2}[a, a] \in \Omega^2(P; \mathfrak{g})^G_{hor}$$

If we have a metric on P, we can define a by the orthogonal complement construction.

Define $\nu \in \Omega^1(P;\mathfrak{g})^G$ via

$$\langle \nu(v), \xi \rangle := g(K^\xi, v)$$

Then $\langle \nu(K^\eta), \xi \rangle = g(K^\eta, K^\xi)$

$$\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\langle S\eta, \xi \rangle} \text{ for some } S = S_p$$

$S_p^{-1}\nu_p = a_p$ is a connection

$$da = d(S^{-1}\nu) = dS^{-1} \cdot \nu + S^{-1} d\nu$$

Need to know $da|_{\mathcal{H}} = F_a|_{\mathcal{H}}$

$$\overset{\shortparallel}{\cancel{dS^{-1} \cdot \nu|_{\mathcal{H}}}} + S^{-1}(d\nu|_{\mathcal{H}})$$

Fourier-transform

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int f(x) e^{ix\xi} dx$$

$$f(b/\lambda) - \frac{1}{2\pi} \int e^{-i \frac{b}{\lambda} \lambda \xi} e^{ix\lambda\xi} f(x) \lambda \, dx d\xi \qquad \begin{matrix} b \rightsquigarrow b/\lambda \\ \xi \rightsquigarrow \xi/\lambda \end{matrix}$$

$x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$

$$f(S^{-1}b) = (2\pi)^{-n} \int e^{-i\langle b, \xi \rangle} e^{i\langle x, S\xi \rangle} f(x) \det S \, d\xi$$

Murad Alim
29.08.18
Göttingen

~~Mapping the UV Theory to the IR Theory~~

Introduction to the u-plane in SW SU(2)-Theory

1. Heuristics
2. Structure of the moduli space
3. Solution of the problem
4. Synthesis

1.1. Setup

$\mathcal{N} = 2$ SUSY gauge theory with the gauge gp $G = SL$
in $\mathbb{R}^{1,3}$ (4d)

Z central charge

Later $\quad Z : \mathcal{M} \longrightarrow \mathrm{Hom}(\Gamma ; \mathbb{C})$, $\quad \Gamma \cong \mathbb{Z}^2$ lattice

moduli space
u-plane

$$u \longmapsto Z_u(\gamma)$$

$$\boxed{\begin{array}{ll} \{Q^1, \bar{Q}^1\} \sim \sigma^\mu \partial_\mu & \\ \{Q^2, \bar{Q}^2\} \sim & \{Q^1, Q^2\} = Z \quad \text{central extension} \end{array}}$$

- A scalar field ϕ in the complexified adjoint rep of SU(2)
- There is a scalar potential $V(\phi) = \frac{1}{g^2} \mathrm{Tr}\left([\phi, \phi^\dagger]\right)^2$

g is the coupling

- $V(\phi) = 0$ allows for example for

$$\phi = \frac{1}{2} a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The Weyl gp of $SU(2)$ acts as $a \to -a$

Use $u = \frac{1}{2} a^2 = \mathrm{Tr}\, \phi^2$

u is a local coordinate on $\mathcal{M}$ = pts of minima of V. $/SU(2)$

1.2 Moduli space

* $\mathcal{M}$ is a Kähler mfld, whose Kähler potential K is specified by a hol. fn $\underset{\overset{\shortparallel}{\mathcal{F}(a)}}{\mathcal{F}}$ called the prepotential

$$K = \mathrm{Im}\left(\frac{\partial \mathcal{F}}{\partial a} \cdot \bar{a}\right)$$, the metric on $\mathcal{M}$

$$ds^2 = \mathrm{Im}\left(\frac{\partial^2 \mathcal{F}}{\partial a^2}\right) da\, d\bar{a}$$

Classically : $\mathcal{F}(a) = \frac{1}{2} \tau_{cl}\, a^2$ (but this is not true generally)

$$\tau_{cl} = \underbrace{\frac{\theta}{2\pi}}_{} + \frac{4\pi}{g^2} i$$

θ is defined only upto an integer shift

1.3 Expected quantum corrections to $\mathcal{F}$

$$\mathcal{F} = \frac{1}{2}\tau_{cl}\, a^2 + \underbrace{\frac{i}{2\pi}\, a^2 \ln\frac{a^2}{\Lambda^2}}_{\mathcal{F}_{1\text{-loop}}} + \sum_{k=1}^{\infty} c_k \left(\frac{\Lambda}{a}\right)^{4k} a^2$$

$$\tau(a) = \frac{\partial^2 \mathcal{F}}{\partial a^2}$$ (here a is a function of u, not necessarily $u = \frac{1}{2}a^2$)

* $\mathrm{Im}\,\tau(a)$ harmonic f-n on $\mathcal{M}$, hence cannot be positive, so the description is valid in a local patch of $\mathcal{M}$ only

* Define $a_D := \dfrac{\partial \mathcal{F}}{\partial a}$, $ds^2 = \mathrm{Im}\,(da_D\, d\bar{a})$

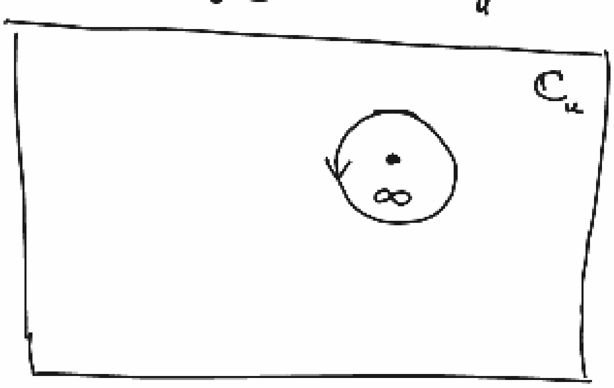
(later $a_D \sim$ mass of a monopole)

2. Structure of the moduli space

2.1 The singularity at infinity

From $\mathcal{F}_{1\text{-loop}} = i\,\dfrac{a^2}{2\pi} \ln\dfrac{a^2}{\Lambda^2}$

$$a_D = \frac{\partial \mathcal{F}}{\partial a} \sim \frac{2ia}{\pi}\ln\frac{a}{\Lambda} + \frac{ia}{\pi}$$



under a loop around $u = \infty$ we have

$\ln u \longrightarrow \ln u + 2\pi i$, $u = \frac{1}{2}a^2$

$\ln a \longrightarrow \ln a + \pi i$

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}}_{\substack{= \\ M_\infty \\ \text{the monodromy matrix}}} \begin{pmatrix} a_D \\ a \end{pmatrix}$$

2.2. <u>Singularity at strong coupling</u>

The monodromies should give a rep of $\pi_1(\mathcal{M})$ in $SL(2;\mathbb{Z})$

Need a non-abelian rep. of $\pi_1(\mathcal{M})$ so we need at least two other sing. pts.

<u>Assumption</u> one additional singularity in $\mathcal{M}$ is caused by a monopole becoming massless.

Assume this happens at $u = u_0$, which can be interpreted

$$a_D(u_0) = 0.$$

(Here we do insist that a_D, as a loc. hol. coord extend over u_0, but it is not clear whether this is a reasonable requirement in general)

At $u = \infty$ (near ∞) $\quad SU(2) \longrightarrow U_e(1)$ (electric)

A

$\prime\prime$ or $u = u_0$ $\quad U_m(1)$ (magnetic)

A_D

τ_D (coupling of $U_m(1)$)

+ monopole (hypermultiplet)

$$\tau_D \sim \frac{-i}{\pi} \ln a_D$$

Here we assume that a_D is a good loc. coord. near u

$$a_D \sim c_0 (u-u_0)$$

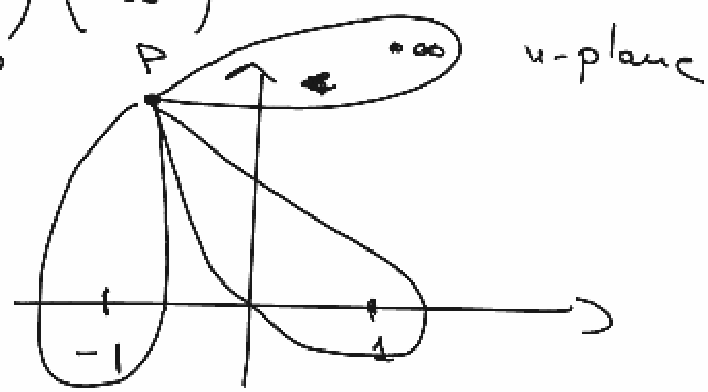
$$\tau_D = \frac{1}{\tau}$$

$$\tau_D = \frac{\partial a}{\partial a_D}$$

$$a \sim a_0 + \frac{i}{\pi}(u-u_0)\ln(u-u_0)$$

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}}_{M_1} \begin{pmatrix} a_D \\ a \end{pmatrix}$$

For now $u_0 = 1$



{ The R-symmetry breaks to a discrete symmetry $u \longrightarrow -u$, so the answer must be invariant under this symmetry }

$$M_1 \cdot M_{-1} = M_\infty^{-1} \Rightarrow M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$$

$\Rightarrow$ an object of charge $(2,1)$ becoming massless.

3. Solution of the problem

$\mathcal{M} = \mathbb{P}^1 \setminus \{-1, 1, \infty\}$ and $\mathbb{Z}_2$-symmetry $u \to -u$

We want a flat $SL(2;\mathbb{Z})$ bundle $V \to \mathcal{M}$ with monodromies M_{-1}, M_1, M_∞ around $-1, 1, \infty$.

Think of $\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix}$ as a holomorphic section of V, possibly multi-valued.

V is determined by the following data:

We want at $u = \infty$

$$a \sim \sqrt{2u}$$
$$a_D \sim i \frac{\sqrt{2u}}{\pi} \ln u$$

at $u = 1$

$$a_D \sim c_0 (u-1)$$
$$a \sim a_0 + \frac{i}{\pi} a_D \ln a_D$$

at $u = -1$ (?) – good loc. coord.

$$a_D + a \sim d_0 (u+1)$$
$$(a_D + a)_D \sim (a_D + a) \ln(a_D$$

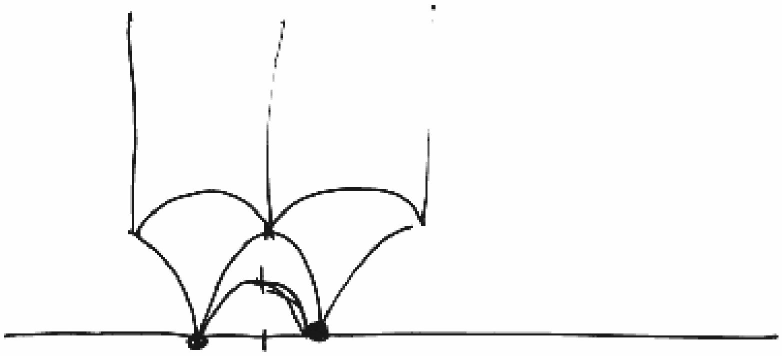
, we want metric on the u-plane to be written as

$$ds^2 = \operatorname{Im} \tau \, |da|^2$$
$$\tau = \frac{da_D/du}{da/du}$$

In particular, we have a rep. $\pi_1(\mathbb{P}^1 \setminus \{-1, 1, \infty\}) \to SL(2;\mathbb{Z})$. The image $\Gamma_{mon} \subset SL(2;\mathbb{Z})$,

$$\Gamma_{mon} = \{ M \in SL(2;\mathbb{Z}) \mid M \equiv \mathrm{Id} \bmod 2 \} =: \Gamma(2)$$

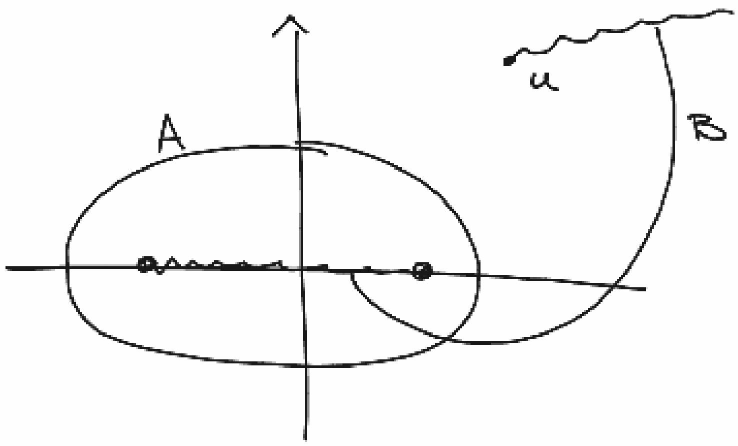
$\Gamma(2)$ is an index 6 subgp



$\exists$ a map $\mathbb{P}^1 \setminus \{-1, 1, \infty\} \xrightarrow{\tau} \underbrace{\Gamma(2) \backslash \mathbb{H}}_{\mathcal{M}}$

moduli space of elliptic curves

$\pi : E \to \mathcal{M}$, $\pi^{-1}(u) = E_u := \{ y^2 = (x-1)(x+1)(x-u) \} \subset \mathbb{C}^2$



$H_1(E_u; \mathbb{Z}) \cong \mathbb{Z}^2$

$H_1(E_u; \mathbb{C}) \cong H_1(E_u; \mathbb{Z}) \otimes \mathbb{C}$

dual to $H_1(E_u; \mathbb{C})$ is $H^1(E_u; \mathbb{C})$, which is spanned by $\lambda_1 = \frac{dx}{y}$ (this a hol. 1-form)

$\lambda_2 = x \frac{dx}{y}$ (this is hol. as well)

Want, $\lambda_{SW} = c_1(u) \lambda_1 + c_2(u) \lambda_2$ s.t.

$$\int_B \lambda_{SW} = a_D \qquad \text{and} \qquad \tau = \frac{da_D/du}{da/du}$$

$$\int_A \lambda_{SW} = a$$

We know $\tau = \dfrac{\int_B \lambda_1}{\int_A \lambda_1}$

Moore : $\lambda_{SW} = \text{const} \cdot (x-u)\dfrac{dx}{y}$ (this is wrong, but ~~[illegible]~~ is nevertheless equivalent)