Exercise sheet 2

Algebraic Topology

October 15, 2024

Exercise 1. Let A be a subspace of the topological space X.

1. Assume that there exists a map $r\colon X \to A$ such that $r|_A$ is the identity map (in that case we call r a *retraction* map and A a *retract* of X). For any $k \in \mathbb{Z}$, let r_k be the induced map of r between $H_k(X)$ to $H_k(A)$. Show that

$$H_k(X) \simeq H_k(A) \oplus \ker r_k.$$

2. Assume that there exists a map $R\colon X \times [0,1] \to X$ such that

   - $R(a,t) = a$ for all $a \in A$ and $t \in [0,1]$,
   - $R(x,0) = x$ for all $x \in X$, and
   - $R(x,1) \in A$ for all $x \in X$

   (in that case we call R a *deformation retraction map* and A a deformation retract of X). Show that $H_k(X)$ is isomorphic to $H_k(A)$ for all $k \in \mathbb{Z}$.

Exercise 2. Let X and Y be topological spaces and let $f_n, g_n\colon S_n(X) \to S_n(Y)$ be chain maps. We say f_n and g_n are *chain homotopic* if there exists a family of maps $P_n\colon C_n(X) \to C_{n+1}(Y)$ such that

$$\partial_{n+1}P_n - P_{n-1}\partial_n = f_n - g_n.$$

We say that the spaces X and Y are *chain homotopic* if there are chain maps $f_n\colon S_n(X) \to S_n(Y)$ and $g_n\colon S_n(Y) \to S_n(X)$ such that $f_n \circ g_n$ and $g_n \circ f_n$ are chain homotopic to the identity maps.

1. Show that if f_n is chain homotopic to g_n, then the induced maps f_*, g_* in homology are equal.

2. Show that if X is chain homotopic to Y, then $H_k(X) \simeq H_k(Y)$ for all $k \in \mathbb{Z}$.

Exercise 3. A singular 1-simplex $\sigma\colon [0,1] \to X$ is called a loop if $\sigma(0) = \sigma(1)$.

1. Prove that a loop is an 1-cycle.
2. Two loops are called freely homotopic if there is a continuous map $F\colon [0,1] \times [0,1] \to X$ such that $F(0,t) = \sigma_0(t)$, $F(1,t) = \sigma_1(t)$, and each $F(s,\ldots)$ is a loop. Show that free homotopy defines an equivalence relation on the sets of loops in X.
3. Show that two freely homotopic loops are homologous, i.e. they define the same element in $H_1(X)$. *Hint: Draw a picture and try to triangularize it.*
4. A 1-chain $\sigma_0 + \ldots + \sigma_{r-1}$ with $\sigma_i(0) = \sigma_{i-1}(1)$ for all $i \in \mathbb{Z}_r$ is called an elementary 1-cycle. Prove that an elementary 1-cycle is a 1-cycle and it is homologous to a loop.
5. Prove that the class of elementary 1-cycles generate $H_1(X)$.