Exercise sheet 3

Algebraic Topology

October 22, 2024

Exercise 1. Let X be a topological space and $A \subseteq X$.

1. Show that if $A = \emptyset$, then $H_k(X, A) = H_k(X)$ for all $k \in \mathbb{Z}$.
2. Assume that X has n connected components. Show that if $A = \{x_0\}$, then

$$H_k(X, A) \simeq \begin{cases} \mathbb{Z}^{n-1} & k = 0 \\ H_k(X) & \text{else.} \end{cases}$$

Exercise 2.

1. Identify S^1 as the equator inside S^2. Calculate $H_k(S^2, S^1)$ for all $k \in \mathbb{Z}$.
2. Let $S^2 \sqcup S^2$ be two disjoint spheres and let p_1, p_2 be the respective north poles. Calculate $H_k(S^2 \sqcup S^2, \{p_1, p_2\})$ for all $k \in \mathbb{Z}$.
3. Compare the above results. Is there a geometric reason why the above homology groups are isomorphic?

Exercise 3. See Exercise 2.35 in the lecture notes: Show that the Bockstein homomorphism ∂ is a group homomorphism.

Exercise 4. See exercise 2.38 in the lecture notes: Let X and Y be topological spaces and let $A \subseteq X$ and $B \subseteq Y$. Let $f\colon X \to Y$ be a map such that $f(A) \subseteq B$. Denote the restriction of f to A as $\hat{f}$. Show that the induced maps in homology make the following diagram commute:

$$\begin{array}{ccc} H_n(X, A) & \xrightarrow{\delta} & H_{n-1}(A) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \hat{f}_*} \\ H_n(Y, B) & \xrightarrow{\delta} & H_{n-1}(B) \end{array}$$

Exercise 5. Let $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ be a short exact sequence of complexes. Show that the induced exact sequence in homology

$$\dots H_k(A) \overset{\alpha}{\to} H_k(B) \overset{\beta}{\to} H_k(C) \overset{\delta}{\to} H_{k-1}(A) \overset{\alpha}{\to} H_{k-1}(B) \to \dots$$

is indeed exact. That is, show that

1. $\operatorname{im} \alpha = \ker \beta$,
2. $\operatorname{im} \beta = \ker \delta$,
3. $\operatorname{im} \delta = \ker \alpha$.