Exercise sheet 4

Algebraic Topology

November 5, 2024

Exercise 1. Let X be a topological space. The **suspension** SX of X is the space $[-1, 1] \times X$, quotiented by the relationship

$$(s, x) \sim (t, y) \Leftrightarrow s = t \text{ and } s, t \in \{\pm 1\}.$$

Show that

- $H_k(X) \simeq H_{k+1}(SX)$ for all $k \geq 1$,

- $H_0(X) \simeq H_1(SX) \oplus \mathbb{Z}$, and

- $H_0(SX) \simeq \mathbb{Z}$.

Exercise 2. Prove Theorem 2.72 from the lecture notes. That is, show that the homology groups of the Klein bottle are

$$H_k(K) \simeq \begin{cases} \mathbb{Z} & \text{for } k = 0, \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{for } k = 1, \\ 0 & \text{else.} \end{cases}$$

Exercise 3. Show that the homology groups of the pinched torus in Figure 1 are

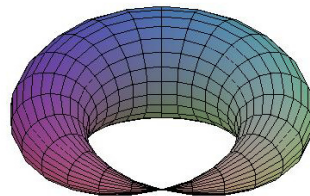$$H_k(K) \simeq \begin{cases} \mathbb{Z} & \text{for } k = 0, 1, 2 \\ 0 & \text{else.} \end{cases}$$

Figure 1: *The pinched torus*

Exercise 4. Let X and Y be path-connected topological spaces, $x \in X$ and $y \in Y$. The **wedge sum** or the one-point union $X \vee Y$ is the disjoint union of X and Y, quotiented by the relationship that identifies x to y. Show that if x and y have contractible neighbourhoods in X and Y respectively,

$$H_k(X \vee Y) = \begin{cases} H_k(X) \oplus H_k(Y) & \text{if } k \geq 1, \\ \mathbb{Z} & \text{if } k = 0. \end{cases}$$