Exercise sheet 8

Algebraic Topology

December 3, 2023

Exercise 1. Recall from Example 3.8, 3.9 and 3.31 that there is a natural way to embed $\mathbb{C}P^n$ and $\mathbb{R}P^n$ inside $\mathbb{C}P^{n+1}$ and $\mathbb{R}P^{n+1}$ respectively. Hence define

$$\mathbb{C}P^\infty = \bigcup_{n\in\mathbb{N}} \mathbb{C}P^n \text{ and } \mathbb{R}P^\infty = \bigcup_{n\in\mathbb{N}} \mathbb{R}P^n.$$

Equip $\mathbb{C}P^\infty$ and $\mathbb{R}P^\infty$ with a CW-complex and show that

$$H_k(\mathbb{C}P^\infty) = \begin{cases} \mathbb{Z} & k \equiv 0 \mod 2 \\ 0 & k \equiv 1 \mod 2, \end{cases}$$

and

$$H_k(\mathbb{R}P^\infty) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}_2 & k \equiv 1 \mod 2, \\ 0 & \text{else.} \end{cases}$$

Exercise 2. Let Σ_g be the compact surface of genus g (See Section 2.11.5). Without using Theorem 2.82, show that the Euler characteristic of Σ_g is

$$\chi(\Sigma_g) = 2 - 2g.$$

Exercise 3. Let $f\colon S^n \to S^n$ be a continuous map for some $n \geq 1$. Viewing S^{n+1} as the suspension of S^n, show that f induces a map $g\colon S^{n+1} \to S^{n+1}$ such that the degree of f equals the degree of g. Use this to create a map of degree 2 on S^2.