Exercise sheet 8

Algebraic Topology

December 10, 2024

Exercise 1. Show that S^n is simply-connected for $n \geq 2$. That is, show that S^n has the same fundamental group of a point, which is $\pi_1(\{pt\}) = \{e\}$.

Exercise 2. See Theorem 4.8. From the isomorphism $\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$ it follows that loops in $X \times \{y_0\}$ and $\{x_0\} \times Y$ represent commuting elements of $\pi_1(X \times Y, (x_0, y_0))$. Construct an explicit homotopy demonstrating this.

Exercise 3. See Proposition 4.6. Show that the fundamental group of a topological space X is Abelian if and only if the map P_ω does not depend on the choice of path ω, but only on its endpoints.

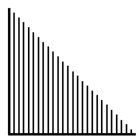

Figure 1: *Example of a contractible space that does not deformation retract to some points.*

Exercise 4. See figure 1.

1. Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0, 1] \times \{0\}$ together with the vertical segments $\{r\} \times [0, 1 - r]$ for r a rational number in $[0, 1]$. Show that X deformation retracts to any point in the segment $[0, 1] \times \{0\}$, but not to any other point.

   Hint: Assume there is a deformation retraction r to another point $x \in X$ and consider the pre-image of some open neighborhood U of x. Show $[0, 1] \times \{x\} \in r^{-1}(U)$. What can you tell about points in the tubular neighborhood of $[0, 1] \times \{x\}$ inside $r^{-1}(U)$?

2. Let Y be the subspace of $\mathbb{R}^2$ that is the union of infinite copies of X arranged as shown in Figure 2. Show that Y is simply-connected, but does not deformation retract to any point.

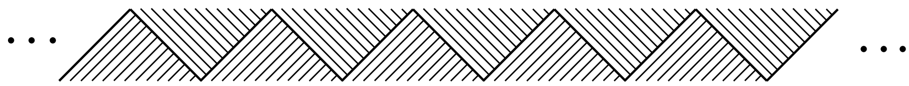

Figure 2: *Example of a contractible space that does not deformation retract to any point.*