# Algebraic topology - Assignment sheet 1

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*For the following exercises, $X$ is a topological space.*

**Exercise 1.** Show that $H_0(X) \simeq \mathbb{Z}^n$ if and only if $X$ has $n$ path-connected components.

**Exercise 2.** Let $X_1, \dots X_n$ be the path-connected components of $X$. Prove that $H_k(X) \simeq \oplus_{i=1}^n H_k(X_i)$ for all $k \in \mathbb{Z}$.

**Definition.** Denote $S_k$ is the set of all $k$-simplices of a topological space $X$. See Figure 4 and 5. A *triangulation* of a topological space $X$ is a choice $C_k \subset S_k$ for each $k$ such that

- Any distinct $\sigma, \tau \in C_k$, $\sigma(\Delta^k)$ and $\tau(\Delta^k)$ are disjoint or their intersection must share a common face.
- The union of all simplices in $C_k$ should cover $X$, and
- $\partial C_k \subset C_{k-1}$.

Given a topological space $X$ with a triangulation $C_k$, one can consider the restriction of the boundary map $\partial_k \colon C_k \to C_{k-1}$. We still have $\partial_{k-1} \circ \partial_k = 0$ and we can define the homology groups

$$H_k^{\Delta}(X) := \frac{\ker \partial_k \colon C_k \to C_{l-1}}{\operatorname{im} \partial_{k+1} \colon C_{k+1} \to C_k}.$$

These homology groups are called the *simplicial homology groups* of $X$.

**Exercise 3.** Compute the simplicial homology of the triangulations shown in Figure 1 and show that they are all isomorphic to

$$H_k \simeq \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & \text{else.} \end{cases}$$

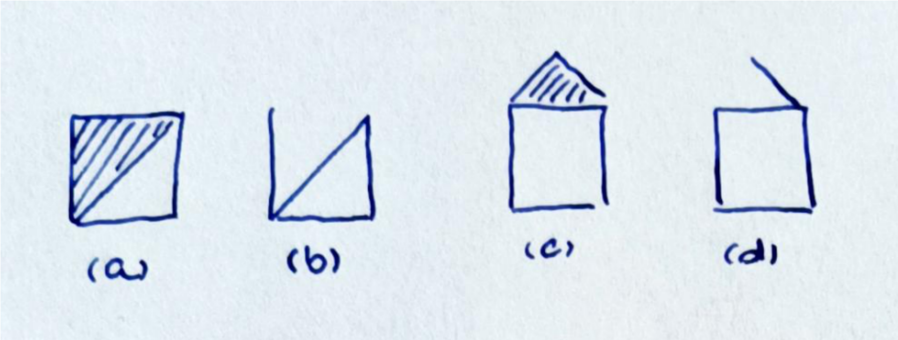


Figure 1: *Note that only in the shaded regions, there are 2-simplices glued inside.*

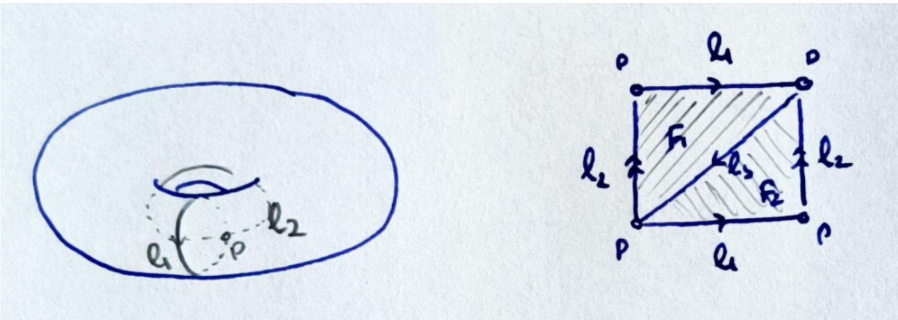


Figure 2: *The torus with a triangulation*

**Exercise 4.**

1. See figure 2. One can construct a torus with a triangulation from a square by identifying the opposite edges, denoted as $l_1$ and $l_2$. Using this triangulation, show that the simplicial homology groups of a torus is

$$H_k \simeq \begin{cases} \mathbb{Z} & k = 0, 2 \\ \mathbb{Z}^2 & k = 1 \\ 0 & \text{else.} \end{cases}$$

2. See figure 3. One can construct a Klein bottle from a square by identifying the opposite edges, but reversing the orientation of one of the sides. Using this triangulation, compute the simplicial homology groups of a Klein bottle.

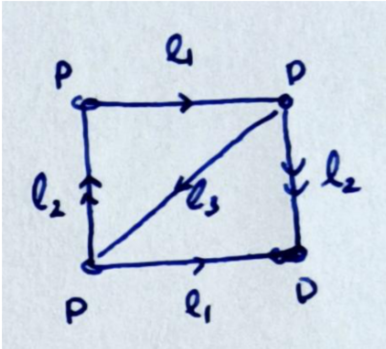


Figure 3: *The Klein bottle with a triangulation*

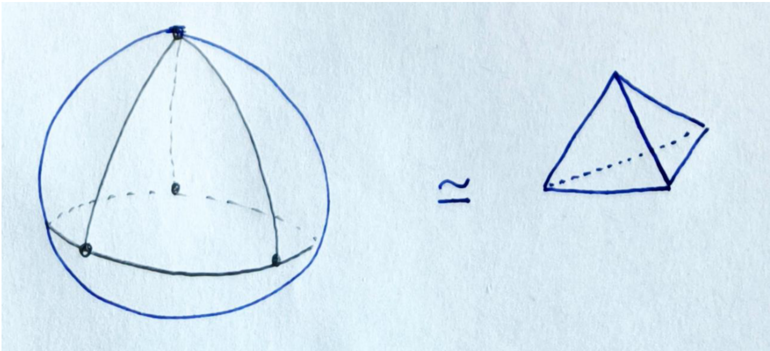


Figure 4: *The 2-sphere with the triangulation of a tetrahedron*

**Exercise 5.**

1. In Figure 4 we have equipped $S^2$ with a triangulation of a tetrahedron. Calculate the homology groups of this space.

2. Consider a different triangulation of $S^2$ with the shown in Figure 5. Compute again the simplicial homology groups and compare your answer with part 1. Do you think the simplicial homology depends on the triangulation of the space?

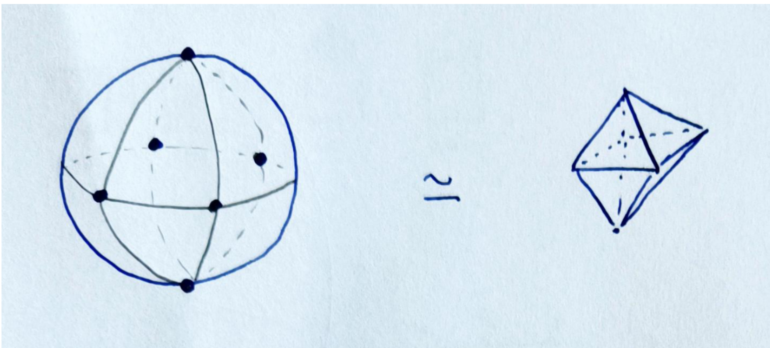


Figure 5: *The 2-sphere with the triangulation of a octahedron*

**Exercise 6.** A singular 1-simplex $\sigma\colon [0,1] \to X$ is called a loop if $\sigma(0) = \sigma(1)$.

1. Prove that a loop is an 1-cycle.
2. A 1-chain $\sigma_0 + \ldots + \sigma_{r-1}$ with $\sigma_i(1) = \sigma_{i-1}(1)$ for all $i \in \mathbb{Z}_r$ is called an elementary 1-cycle. Prove that an elementary 1-cycle is a 1-cycle and it is homologous to a loop.
3. Prove that the class of elementary 1-cycles generate $H_1(X)$.
4. Two loops are called freely homotopic if there is a continuous map $F\colon [0,1] \times [0,1] \to X$ such that $F(0,t) = \sigma_0(t)$, $F(1,t) = \sigma_1(t)$, and each $F(s,\ldots)$ is a loop. Show that free homotopy defines an equivalence relation on the sets of loops in $X$.
5. Show that two freely homotopic loops are homologous, i.e. they define the same element in $H_1(X)$.
6. Assume that $X$ is path connected. Conclude that the map

$$\frac{\{\text{Loops in } X\}}{\{\text{Free homotopy equivalence}\}} \to H_1(X)$$

is surjective.

*Actually, in chapter 4 we will show that these equivalence classes can be given a group structure. In Theorem 4.9 we will see that $H_1(X)$ is isomorphic to the abelianization of this group.*