MATH-F310: Differential Geometry I
- Assignment 10 -

Vector fields

1. (Hairy Ball theorem) The hairy ball theorem states that there exists a (smooth) nowhere zero vector field on S^k if and only if k is odd. The goal of this exercise is to prove this theorem.

   (a) ♢ [1] Show that there exists a vector field v on S^{2k+1} having no zeros. *(Hint: Use the fact that $S^{2k+1} \subset \mathbb{C}^{k+1}$.)* Prove in addition that the tangent bundle of S^3 is diffeomorphic to $S^3 \times \mathbb{R}^3$.

   (b) ♡ Now suppose S^k has a nowhere zero vector field v. Show that we may assume that v has unit length.

   (c) $\forall t \in \mathbb{R}$, let $f_t : S^k \to S^k$ be defined by

   $$f_t(x) = \cos(t)x + \sin(t)v(x).$$

   Prove that f_t defines a smooth homotopy between the identity and the antipodal map.

   (d) Compute the determinant of the differential of the identity and the antipodal map (at any point).

   (e) Consider the map $F : S^k \times S^1 \to S^k, F(x, e^{it}) := f_t(x)$. (Note that it is well defined and smooth). Use Sard's theorem to show that there exists $y \in S^k$ such that $F^{-1}(y)$ is a smooth, compact 1-dimensionnal manifold of $S^k \times S^1$. Hence $F^{-1}(y)$ is a union of circles.

   (f) † Understand why no such circle can intersect twice the spheres $S^k \times \{e^{i0}\}$ and $S^k \times \{e^{i\pi}\}$. Therefore, understand why there is a continuous path

[1] Exercises marked by a ♢ will be done in class (if time permits).
Exercises marked by a ♡ should be done at home.
Exercises marked by a † are extra exercises.

$t \in [0,\pi] \to x_t \in S^k$ such that $f_t(x_t) = y$. Maybe up to some homotopy of F, we may assume that $\forall t \in [0,\pi]$, $df_t|_{x_t}$ is an isomorphism.

(g) Find a contradiction if k is even.

2. $\diamondsuit$ A smooth vector field v on a smooth manifold M is said to be complete, if for each $p \in X$, the maximal integral curve of v through p, has domain equal to $\mathbb{R}$. Determine which of the following vector fields are complete:

(a) $v(x,y) = (1,0), M = \mathbb{R}^2$
(b) $v(x,y) = (1,0), M = \mathbb{R}^2\backslash\{(0,0)\}$
(c) $v(x,y) = (-y,x), M = \mathbb{R}^2\backslash\{(0,0)\}$
(d) $v(x,y) = \left(1+x^2, 0\right), M = \mathbb{R}^2$

3. $\heartsuit$ Let M be a smooth manifold and let $X, Y \in \mathfrak{X}(M)$ be smooth vector fields. Recall that tangent vectors act on smooth functions $f \in C^\infty(M)$ ($X_p(f) = df_p(X)$). Hence $X(f)$ is a function from $M \to \mathbb{R}$, $p \mapsto X_p(f)$. Define the operator $[X,Y]$ by its action on smooth functions

$$[X,Y]_p(f) := X_p(Y(f)) - Y_p(X(f)).$$

Show that $[X,Y]_p$ is a tangent vector (show that its action is linear and satisfies the Leibniz rule). Compute the components of $[X,Y]$ in the basis associated to coordinate charts. Prove that $[X,Y]$ is a smooth vector field.

4. $\heartsuit$ Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Define the vector field X^A on $\mathbb{R}^n$ by

$$X^A_p := A(p) \in \mathbb{R}^n \cong T_p\mathbb{R}^n.$$

Compute the flow φ^A_t of X^A and prove that forall t, it is a linear map.

(a) Compute the flow in the particular case where $A : \mathbb{C}^n \to \mathbb{C}^n$ is diagonal multiplication by i and when $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ acting on $\mathbb{R}^2$.

(b) Suppose A is skew symmetric, then show that φ^A_t is an isometry. In other words, if A is in $\mathfrak{so}(n)$, then $t \mapsto \varphi^A_t$ is a curve in $\mathrm{SO}(n)$. This is a general phenomenon relating a Lie algebra and its associated Lie group.