MATH-F310: Differential Geometry I
- Assignment 11 -

Exam preparation

1. **Hyperbolic space:**
   Let $H := \{(t,x,y) \in \mathbb{R}^3 \mid -t^2 + x^2 + y^2 = -1, t > 0\}$.
   (a) Prove that H is a surface and describe its tangent space at a generic point $p \in H$.
   (b) Let $Q : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the following bilinear form:
   $$Q\left((t_1,x_1,y_1),(t_2,x_2,y_2)\right) := -t_1t_2 + x_1x_2 + y_1y_2.$$
   For any $\lambda \in \mathbb{R}$ and any $\theta \in [0, 2\pi)$, define the boost and rotation matrices by
   $$F_\lambda := \begin{pmatrix} \cosh(\lambda) & \sinh(\lambda) & 0 \\ \sinh(\lambda) & \cosh(\lambda) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_\theta := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}.$$
   Prove that both F_λ and R_θ preserve Q i.e. show that
   $$\forall X \in \mathbb{R}^3, \quad Q(X,X) = Q(F_\lambda X, F_\lambda X) \quad \text{and} \quad Q(X,X) = Q(R_\theta X, R_\theta X).$$
   (c) Show that the restriction of F_λ to H defines a smooth map $f_\lambda : H \to H$. Compute the differential of f_λ.
   (d) Prove that any point $p \in H$ can be obtained from $p_0 = (1,0,0) \in H$ by a boost and a rotation i.e. that there exist $\lambda \in \mathbb{R}$ and $\theta \in [0, 2\pi)$ such that
   $$p = R_\theta \circ f_\lambda(p_0). \tag{1}$$
   (e) For $p \in H$, let $g_p = Q|_{T_pH \times T_pH}$ be the restriction of Q to T_pH. Prove that g is a Riemannian metric in the following way:

i. Prove that at $p_0 = (1,0,0)$, g_{p_0} is positive definite.

ii. Prove that $\forall X \in T_{p_0}H$

$$g_{p_0}(X,X) = g_{R_\theta \circ f_\lambda(p_0)}\left(d(R_\theta \circ f_\lambda)X, d(R_\theta \circ f_\lambda)X\right).$$

iii. Conclude.

2. **Short exercise on the Torus:**
Let

$$T_{r,R} := \left\{ \left(\sqrt{x^2+y^2} - R\right)^2 + z^2 = r^2 \right\}$$

be a torus in $\mathbb{R}^3$, $R > r$. Let $f : T_{r,R} \to \mathbb{R}$ be the restriction of the function $\mathbb{R}^3 \to \mathbb{R}, (x,y,z) \mapsto z$. Compute the critical points of f and the Gauss curvature K of $T_{r,R}$ at those critical points. (Hint: You can answer these questions without a lot of computations.)

3. **Klein bottle:**
Let $T = S^1 \times S^1$ be the Torus.

(a) Prove that T admits an oriented atlas.

(b) Consider the map $f : T \to T$ given by

$$f(z_1, z_2) := (-z_1, \overline{z_2}).$$

Prove that f is a local diffemorphism and that it reverses the orientation. (Either in the sense of exercise sheet 6, or in the following sense: Let $(U, \varphi = (x_1, x_2))$ be a chart near p and $(W, \chi = (y_1, y_2))$ be a chart near $f(p)$. Suppose that both chart come from the same oriented atlas, and let $J_f(p)$ be the matrix representation of df_p in the bases $\{\partial_{x_1}, \partial_{x_2}\}$ of T_pT and $\{\partial_{y_1}, \partial_{y_2}\}$ of $T_{f(p)}T$. Then f reverses the orientation if and only if $\forall p \in T \det J_f(p) < 0$.)

(c) Consider the Klein bottle $K := T/\sim$ where $p \sim f(p)$ and the quotient map $q : T \to K$. Construct an atlas on K for which q is a local diffeomorphism.

(d) Prove that this atlas on K is not oriented.

4. **Cotangent bundle and Hamiltonian mechanics:**
Part 1, abstract: Let M^n be a smooth manifold with smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$. We denote by $x_1, \dots, x_n$ the coordinates on $\mathbb{R}^n$ and identify them with functions $\mathbb{R}^n \to \mathbb{R}$.

(a) Prove that for each $q \in U_\alpha$, $\{d(x_i \circ \varphi_\alpha)_q\}_i$ forms a basis of T_q^*M.

(b) Construct a smooth atlas of T^*M and prove that the transition functions are smooth.

Part 2, concrete: Now we specialise to $M = \mathbb{R}$ with global coordinate x. An element of $T^*\mathbb{R}$ is a point $q \in \mathbb{R}$ together with an element $p\,dx_q \in T_q^*\mathbb{R}$. So

$$T^*\mathbb{R} \cong \mathbb{R}^2,\ (q, p\,dx_q) \mapsto (x(q), p).$$

Let $V : \mathbb{R} \to \mathbb{R}$ be a smooth function and define the vector field X on $T^*\mathbb{R}$ given by

$$X(x,p) := p\,\partial_x - \frac{\partial V}{\partial x}\partial_p.$$

(Recall that ∂_x and ∂_p are the speed along the curves $t \mapsto (x+t,p)$ and $t \mapsto (x, p+t)$.)

(c) Let $\tilde{\gamma} : \mathbb{R} \to T^*\mathbb{R}$ be an integral curve of X. Write down the equations for $\tilde{\gamma}$ and prove that $\gamma := \pi \circ \tilde{\gamma}$ ($\pi : T^*\mathbb{R} \to \mathbb{R}$ is the projection) solves the equation of motion $\gamma'' = -\nabla V$.

(d) Prove that $\tilde{\gamma}$ stays on the same level set of the energy function

$$E : T^*\mathbb{R} \to R,\ E(x,p) = V(x) + \frac{1}{2}p^2.$$

(e) Suppose $V(x) = x^2/2$. Draw the level sets of E and find the integral curves of X. $\pi \circ \tilde{\gamma}$ is called the harmonic oscillator.