MATH-F310: Differential Geometry I
- Assignment 3 -

Smooth maps

1. $\diamondsuit$[1] For $R > r$, let

$$T := \left\{ \left(\sqrt{x^2 + y^2} - R \right)^2 + z^2 = r^2 \right\}$$

be the torus in $\mathbb{R}^3$. Prove that T is homeomorphic to $S^1 \times S^1$. Regarding S^1 as a subset of $\mathbb{C}$, we define a map

$$f : S^1 \times S^1 \to S^1 \times S^1, \qquad (z_1, z_2) \mapsto (z_1 z_2, z_1 \overline{z_2}).$$

Find a homeomorphism $h : T \to S^1 \times S^1$ such that $h^{-1} \circ f \circ h$ is smooth.

2. $\diamondsuit$ Construct a smooth map $S^2 \to S^2$ by using the map $f : \mathbb{C} \to \mathbb{C}, z \mapsto z^2$ and the stereographic projection. Compute the coordinate representations of this map and prove that it is smooth.

3. $\diamondsuit$ A polynomial $p(x)$ of n variables $(x_1, \dots, x_n) = x$ is called homogeneous of degree $k \in \mathbb{N}$ if $\forall t \in \mathbb{R}$

$$p(tx) = t^k p(x).$$

Let q be a homogeneous polynomial of degree 5 in three variables. Prove that the set of points $x \in \mathbb{R}^3$ such that $q(x) = a$ is a smooth surface provided that $a \neq 0$.
Hint : Use Euler's identity for homogeneous polynomials: $\sum_{i=1}^{n} x \frac{\partial p}{\partial x_i} = kp$.

4. $\spadesuit$ Prove that a connected surface is path-connected.

5. $\spadesuit$ Recall $\psi_N : S^2 \setminus \{N\} \to \mathbb{C}$ the stereographic projection with respect to the north pole N. Let S be the south pole. We define a map on $f : S^2 \to S^2$ in

[1] Exercises marked by a $\diamondsuit$ will be done in class (if time permits).
Exercises marked by a $\spadesuit$ are to prepare at home for the first test (8th of November).
Exercises marked by a † are extra exercises.

the following way:

$$x \in S^2 \backslash \{N, S\} \mapsto \psi_N^{-1} \circ (z \mapsto \frac{1}{z}) \circ \psi_N(x)$$
$$f(N) = S \text{ and } f(S) = N.$$

Prove that f is smooth.

6. † Let $a \in \mathbb{R}$ and let $\gamma_a : \mathbb{R} \to S^1 \times S^1$, $t \mapsto (e^{i2\pi t}, e^{i2\pi a t})$ be a curve on the Torus (that you may embedd in $\mathbb{R}^3$ using the map $h^{-1} : S^1 \times S^1 \to T$ from exercise 1). Show that γ_a is smooth and make a drawing of it's image for some values of a. When does γ_a loops back on itself ?