MATH-F310: Differential Geometry I
- Assignment 4 -

Tangent space

1. ♢ [1] Let $S \subset \mathbb{R}^3$ be a surface and $\psi : V \subset \mathbb{R}^2 \to S$ be a local parametrisation. Denote by (u, v) denote the canonical coordinates of $\mathbb{R}^2$.
   (a) Show that $\{D_u\psi, D_v\psi\}$ is a basis of TS on $\psi(V)$.
   (b) For a curve $\gamma : (-1, 1) \to S$ with $\gamma(0) \in \psi(V)$, write down $\dot{\gamma}(0)$ in terms of that basis.
   (c) For $p \in \psi(V)$ and $X = X_u D_u\psi + X_v D_v\psi \in T_pS$, find a curve γ such that $\dot{\gamma}(0) = X$.
   (d) Let $f : S \to \mathbb{R}$ be a smooth function. Describe df in terms of the basis $\{D_u\psi, D_v\psi\}$ (as a matrix for example).
   (e) Let S_1, S_2 be two surfaces and let $F : S_1 \to S_2$ be a smooth map. Compute dF in terms of bases given by local parametrisations.
2. ♢ Describe the tangent space at a point of the sphere S^2 in two ways : Using local charts and using the fact that S^2 is a level set. Compute the differential of the antipodal map: $S^2 \to S^2, x \mapsto -x$ at a point on the sphere.
3. ♢ *Cotangent space :* The dual of the tangent space of a surface S at the point p is called the cotangent space and is denoted by T_p^*S. Show that for any $f \in C^\infty(S)$, $d_pf \in T_p^*S$. Let $\psi : V \to S$ be a local parametrisation of S near p and let (u, v) be coordinates on $\mathbb{R}^2$. Write $\tilde{u} = u \circ \psi^{-1}$ and $\tilde{v} = v \circ \psi^{-1}$. Show that $d\tilde{u}$ and $d\tilde{v}$ form a local basis for T^*S and for $f \in C^\infty(S)$, write d_pf in that basis.
   Note : In textbooks, people often identify u and $\tilde{u}$ by abuse of notation. Indeed, there is only one obvious meaning of the statement $d_pu \in T_p^*S$.

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[1] Exercises marked by a ♢ will be done in class (if time permits).
Exercises marked by a ♠ are to prepare at home for the first test (8th of November).
Exercises marked by a † are extra exercises.

4. ♠ Describe the tangent spaces of:
   (a) The hyperboloid $H = \{x^2 + y^2 - z^2 = 1\}$.
   (b) The torus $T := \left\{ \left(\sqrt{x^2 + y^2} - R \right)^2 + z^2 = r^2 \right\}$, $R > r$.

5. ♠ Let S^2 be the unit sphere in $\mathbb{R}^3$. Show that at any point $(x, y, z) \in S^2$, the vector $(-y, x, 0)$ is tangent to the sphere. For $p \in S^2$, let $X_p := (-y(p), x(p), 0) \in T_pS^2$. Find a curve $\gamma : \mathbb{R} \to S^2$ such that for all $t \in \mathbb{R}$,

$$\dot{\gamma}(t) = X_{\gamma(t)}.$$

   (It may be useful to make a drawing of the vector field X). γ is called an integral curve of X.

6. † *Tangent vectors as derivations :* Let $S \subset \mathbb{R}^3$ be a surface and denote by $C^\infty(S)$ the space of smooth functions $f : S \to \mathbb{R}$.
   (a) Let $X \in T_pS$ be represented by the initial derivative of a curve $\gamma : (-1, 1) \to S$ (i.e. $X = \dot{\gamma}(0)$). Define the action of X on $C^\infty(S)$ by

$$X(f) := \frac{d}{dt} f \circ \gamma(t)|_{t=0} = df(X) \quad \forall f \in C^\infty(S).$$

   Show that this action is well defined (it doesn't depend on the curve representing X).
   (b) Prove that the Leibniz rule holds i.e. for any $f, g \in C^\infty(S)$,

$$X(fg) = X(f)g(p) + f(p)X(g). \tag{1}$$

   (c) Let $\psi : V \to S$ be a local parametrisation and (u, v) be coordinates on $\mathbb{R}^2$. Show that the action $D_u\psi$ is given by

$$D_u\psi|_p(f) = \frac{\partial}{\partial u}(f \circ \psi)|_{\psi^{-1}(p)} \quad \forall f \in C^\infty(S).$$

   Note : In many textbooks, you will see the notation $\frac{\partial}{\partial u}|_p \in T_pS$ to mean $D_u\psi|_p$.
   (d) For any $p \in S$ let

$$\mathcal{D}_p := \{X : C^\infty(S) \to \mathbb{R} \,|\, X \text{ satisfies (1)}\}$$

   be the space of operators satisfying the Leibniz rule at p. We have shown that

$$\Phi : T_pS \to \mathcal{D}_p,\ V \mapsto (f \mapsto V(f))$$

   is well defined. Show that $\mathcal{D}_p$ is a vector space and that Φ is injective.

(e) The rest of the exercise aims to prove that Φ is an isomorphism of vector spaces.

i. Let $\varphi \in C^\infty(\mathbb{R}^2)$. Prove that there exist functions φ_1 and φ_2 such that

$$\varphi = \varphi(0) + u\varphi_1 + v\varphi_2.$$

(Hint : Write $\varphi(x) = \varphi(0) + \int_0^1 \frac{d}{dt}\varphi(tx)\,dt$.) What is $\varphi_i(0)$?

ii. Let $X \in \mathcal{D}_p$ and $f \in C^\infty(S)$. By composing with a translation if neccessary, we may assume that $\psi^{-1}(p) = 0 \in \mathbb{R}^2$. Show that there exist local functions $f_1, f_2 : \psi(V) \to \mathbb{R}$ such that

$$f|_{\psi(V)} = f(p) + (u \circ \psi^{-1})f_1 + (v \circ \psi^{-1})f_2.$$

iii. Show that $X \in \Phi(T_pS)$.