MATH-F310: Differential Geometry I
- Assignment 6 -

Orientation and integration

1. $\diamondsuit$ [1] Suppose that a surface S is a union $S = S_1 \cup S_2$ where S_1 and S_2 are two orientable surfaces such that $S_1 \cap S_2$ is connected. Prove that S is also orientable. Is it true if there are 3 surfaces (with connected pairwise intersection)?

2. $\diamondsuit$ Define vol : $(\mathbb{R}^n)^n \to \mathbb{R}$ by

$$\mathrm{vol}(v_1, \dots, v_n) := \det(v_j^i)_{1 \leq i,j \leq n}$$

where v_j^i are the coordinates of the vector v_j.

   (a) Prove that vol is skew-symmetric in its arguments and that $\mathrm{vol}(v_1, \dots, v_n)$ is the signed volume of the parallelepiped whose sides are $v_1, \dots, v_n$.

   (b) Suppose $P \subset \mathbb{R}^3$ is a 2-dimensional plane (think $P = T_pS$), then for $a, b \in P$ and $n \perp P$, prove that

$$\langle a \times b, n \rangle = \mathrm{vol}(a, b, n).$$

   (c) † Let S_1, S_2 be two oriented surfaces with unit normal vector fields N_1 and N_2 respectively. We say that a map $f : S_1 \to S_2$ *preserves the orientation* if

$$\mathrm{vol}\,(d_pf(a), d_pf(b), N_2(f(p))) > 0 \quad \forall p \in S_1$$

   whenever $a, b \in T_pS$ form a basis satisfying $\mathrm{vol}(a, b, N_1(p)) > 0$ (we call such basis *oriented*). We say that f *reverses the orientation* if

$$\mathrm{vol}\,(d_pf(a), d_pf(b), N_2(f(p))) < 0 \quad \forall p \in S_1.$$

   Find examples of orientation reversing and orientation preserving maps from S^2 to itself.

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[1] Exercises marked by a $\diamondsuit$ will be done in class (if time permits).
Exercises marked by a $\clubsuit$ are to prepare at home for the second test.
Exercises marked by a † are extra exercises.

3. $\diamondsuit$ *(Reminder on Stokes theorem)* Let $Z : (x, y, z) \mapsto (0, 0, 1)$ be the constant vector field in the direction of z. Compute the flux of Z on the hemisphere $S^+ := \{x^2 + y^2 + z^2 = 1, z > 0\}$ in three different ways:
   (a) By a direct computation.
   (b) By proving that the flux is equal to the one through the unit disk $D = \{x^2 + y^2 < 1, z = 0\}$.
   (c) By finding a vector field Y such that $Z = \nabla \times Y$.
4. $\clubsuit$ Let $f : S_1 \to S_2$ be a local diffeomorphism between two surfaces S_1 and S_2 with S_2 orientable. Let N_2 be a unit normal field on S_2. We define a map $N_1 : S_1 \to \mathbb{R}^3$ as follows: if $p \in S_1$, we put

$$N_1(p) = \frac{a \times b}{|a \times b|}$$

   where a, b form a basis of T_pS_1 satisfying

$$\mathrm{vol}\,(d_pf(a), d_pf(b), N_2(f(p))) > 0.$$

   Show that N_1 is a unit normal field on S_1 and consequently S_1 is also orientable.
5. $\clubsuit$ Let $T = \{(\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2\}$ be a torus in $\mathbb{R}^3$ $(R > r)$.
   (a) Compute the area of the torus T.
   (b) Compute the integral of the function $f : (x, y, z) \mapsto z^2 + 1$ on T.
   (c) Compute the flux of the vector field $X : (x, y, z) \mapsto (-z, 0, x)$.
6. $\dagger$ *Orientable fillings of a knot:* Let $K \subset \mathbb{R}^3$ be a knot, that is K is the image of an injective curve $\gamma : [0, 1] \to \mathbb{R}^3$ which is closed ($\gamma(0) = \gamma(1)$) and regular ($\dot\gamma(t) \neq 0 \,\forall t \in [0, 1]$). Two knots are called equivalent if we can wiggle one into the other.
   A natural question arise : Can we find a surface S whose boundary $\partial S := \overline{S} \backslash S$ is (equivalent to) K? The following algorithm shows you how to do it.

   Start with a diagram on the knot i.e. a drawing of K where at most two lines can cross at the same point. Then resolve the crossings by cutting the strands, and connecting the incoming strand of A with the outgoing strand of B, and vice versa. (see Figure 2[2]) You end up with a bunch of circles, that you can fill

[2]These pictures come from the nice article "Visualization of Seifert surfaces" by Van Wijk and Cohen.

in by disks (maybe separate overlapping disks by playing with their height). Finally, join the circles by twisted bands that mimic the crossings you had originally.

Exercise : Prove that this surface is always orientable. Find orientable and non-orientable fillings of the unknot and of the trefoil.

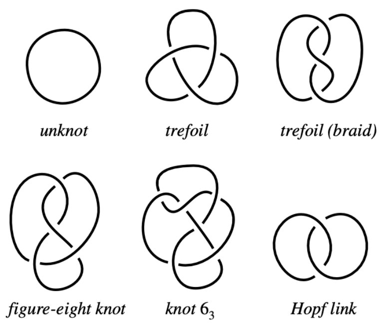

Figure 1: A few knot diagrams.

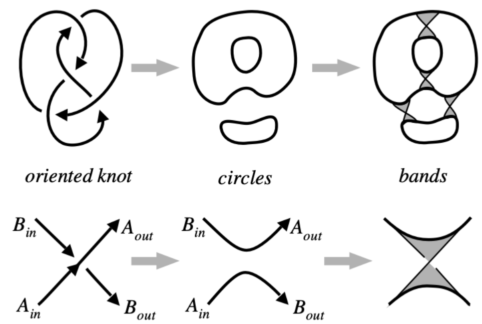

Figure 2: Seifert algorithm. Resolve the crossings, fill the circles in, add twisted bands.