MATH-F310: Differential Geometry I
- Assignment 7 -

Curvature

1. $\diamondsuit$ [1] Let $S \subset \mathbb{R}^3$ be an oriented surface and $n : S \to S^2$ be its Gauss map. The direction of the eigen-vectors of $dn_p : T_pS \to T_pS$ are called the *principal directions* while the eigen-values are called the *principal curvatures* (note that they depend on the sign of the Gauss map). Show that the principal directions are orthogonal.
   Prove that if S is a surface of revolution (i.e. in cylindrical coordinates,
   $$S = \{(\rho \cos\theta, \rho \sin\theta, z) \in \mathbb{R}^3 \mid \rho = f(z)\}$$
   with $f : \mathbb{R} \to \mathbb{R}$ is positive), then the meridians ($\{\theta = cst\}$) and parallels ($\{z = cst\}$) define principal directions.
   Deduce that the cylinder and the cone are flat (i.e. they satisfy $K = 0$ everywhere).

2. $\diamondsuit$ Let $\gamma : I \to S$ be a smooth curve where I is an open interval. Suppose that γ is regular (i.e. $\gamma'(t) \neq 0 \ \forall t \in I$).
   (a) Let $\gamma \circ \varphi$ be a reparametrisation of γ ($\varphi : I' \to I$ is an increasing function). Understand how the acceleration $\gamma''(t)$ changes under such reparametrisation.
   Define the curvature of γ as
   $$k^\gamma(\gamma(t)) := (\gamma''(t))^\perp / |\gamma'(t)|^2,$$
   where $\perp$ denotes the component orthogonal to $\gamma'(t)$. Prove that k^γ doesn't depend on the parametrisation of γ.
   <u>Note:</u> If γ draws a circle of radius R, then $k^\gamma = 1/R$.

[1] Exercises marked by a $\diamondsuit$ will be done in class (if time permits).
Exercises marked by a $\clubsuit$ are to prepare at home for the second test.
Exercises marked by a † are extra exercises.

(b) Prove that there exists an arc length parametrisation of γ, i.e. a parametrisation $\gamma(s)$ for which $|\gamma'(s)| = 1 \, \forall s$. Prove then that

$$k^{\gamma}(\gamma(s)) = \gamma''(s).$$

(c) Decompose the curvature as $k^{\gamma} = k_S^{\gamma} + k_N^{\gamma}$ where $k_S^{\gamma} \in TS$ and $k_N^{\gamma} \perp TS$. Prove that

$$k_N^{\gamma}(\gamma(t)) = -\left\langle \frac{\gamma'(t)}{|\gamma'(t)|}, dn\left(\frac{\gamma'(t)}{|\gamma'(t)|}\right)\right\rangle n(\gamma(t))$$

where n is the Gauss map.
Remark: In particular, $k_N^{\gamma}(\gamma(t))$ only depends on the point $\gamma(t)$ and on the direction of $\gamma'(t)$ but not *on the full trajectory of* γ. Therefore, $k_N^{\gamma}(p)$ computes the curvature of S in the direction of γ' at the point p while k_S^{γ} computes the curvature of γ in S. The generalisations of straight lines, called geodesics, are the curves γ that satisfy $k_S^{\gamma} = 0$. "They do not turn in S."

(d) Fix $p \in S$ and denote by $k_1(p), k_2(p)$ the maximal and minimal values of $\gamma \mapsto \langle k_N^{\gamma}(p), n(p)\rangle$. Prove that

$$K(p) = k_1(p)k_2(p) \quad \text{and} \quad H(p) = \frac{1}{2}(k_1(p) + k_2(p)).$$

3. ♣ Suppose S is a surface of revolution. Compute the principal curvatures (eigen-values of dn) at each point of S. Prove that the Catenoid

$$C_a := \{(\rho\cos\theta, \rho\sin\theta, z) \in \mathbb{R}^3 \,|\, \rho(z) = a\cosh(z/a)\}$$

is minimal (i.e. $H = 0$ everywhere) for any $a > 0$.

4. ♣ Show that an orientable connected surface whose Gauss and mean curvatures are identically zero must be an open subset of a plane.

5. † Compute the Gauss and mean curature at every point of the Torus $T := \left\{\left(\sqrt{x^2+y^2} - R\right)^2 + z^2 = r^2\right\}$, $R > r$.