MATH-F310: Differential Geometry I
- Assignment 8 -

Manifolds

1. $\diamondsuit$ [1] Let $\mathbb{C}P^n$ be the set of all complex lines in $\mathbb{C}^{n+1}$. Prove that $\mathbb{C}P^n$ is a smooth manifold. Prove that $\mathbb{C}P^1$ is diffeomorphic to S^2.
2. $\diamondsuit$ One can define $\mathbb{R}P^2 := S^2/\sim$, where $(x, y, z) \sim (-x, -y, -z)$.
   (a) Show that $\mathbb{R}P^2$ is a smooth manifold.
   (b) Define a map $f : S^2 \to \mathbb{R}^4$, by
   $$f(x, y, z) := \big(xy, xz, y^2 - z^2, 2yz\big)$$
   Show that it is smooth. Moreover prove that it induces a smooth map $\tilde{f} : \mathbb{R}P^2 \to \mathbb{R}^4$.
   (c) Prove that the differential of $\tilde{f}$ is injective at all points of $\mathbb{R}P^2$.
3. $\diamondsuit$ Let $SU(2) := U(2) \cap SL(2, \mathbb{C})$ be unitary 2×2 matrices with determinant one. Show that the map $f : S^3 \subset \mathbb{C}^2 \to SU(2)$,
   $$f(\alpha, \beta) = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$$
   is a homeomorphism ($\alpha, \beta \in \mathbb{C}$). And hence we can construct charts on $SU(2)$ to prove that it's a smooth manifold of dimension 3 .
4. $\clubsuit$ Let V be a vector space over $\mathbb{R}$, show that the general linear group
   $$\mathrm{GL}(V) := \{A : V \to V \mid A \text{ is linear and invertible}\}$$
   is a smooth manifold. What is its dimension?

[1]Exercises marked by a $\diamondsuit$ will be done in class (if time permits).
Exercises marked by a $\clubsuit$ are to prepare at home for the second test.
Exercises marked by a † are extra exercises.

5. ♣ Consider the function

$$f : \mathbb{R}P^2 \to \mathbb{R}, \quad f([x : y : z]) = \frac{xy + yz + xz}{x^2 + y^2 + z^2}$$

Show that f is smooth and find all critical points of f.

6. † (*Canonical line bundle :*) Let M be the space of points in $\mathbb{R}^{n+1}$ together with a line through that point. Formally,

$$M := \{(p, l) \in \mathbb{R}^{n+1} \times \mathbb{R}P^n \,|\, p \in l\}.$$

Prove that M is a smooth manifold. Consider $\pi : M \to \mathbb{R}P^n$ defined by $\pi(p, l) = l$. Prove that π is a surjection and that for any $l \in \mathbb{R}P^n$, $\pi^{-1}(l)$ is a one dimensionnal submanifold of M that we may identify with l.