MATH-F310: Differential Geometry I
- Assignment 9 -

Tangent space and submanifolds

1. ♢[1] (*Tangent vectors as derivations*[2]*:*) Let M be a manifold of dimension n with $p \in M$.

   (a) Let $X \in T_pM$ be represented by the initial derivative of a curve $\gamma : (-1, 1) \to M$ (i.e. $X = [\gamma]$). Define the action of X on $C^\infty(M)$ by

   $$X(f) := \frac{d}{dt} f \circ \gamma(t)|_{t=0} = df(X) \quad \forall f \in C^\infty(M).$$

   Show that this action is well defined (it doesn't depend on the curve representing X).

   (b) Prove that the Leibniz rule holds i.e. for any $f, g \in C^\infty(M)$,

   $$X(fg) = X(f)g(p) + f(p)X(g). \tag{1}$$

   (c) Let $\varphi : U \subset M \to \mathbb{R}^n$ be a local chart and x^i be coordinates on $\mathbb{R}^n$. Fix $p \in U$ and let $\gamma_i : (-\epsilon, \epsilon) \to U$ be the curve starting at p and along the x^i coordinate in the chart i.e. defined by

   $$\varphi \circ \gamma_i(t) = \varphi(p) + (0, ..., \underbrace{t}_{i}, ...0).$$

   Show that the action of $[\gamma_i]$ is given by

   $$[\gamma_i](f) = \frac{\partial}{\partial x^i}(f \circ \varphi^{-1})|_{\varphi(p)} \quad \forall f \in C^\infty(M).$$

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[1]Exercises marked by a ♢ will be done in class (if time permits).
Exercises marked by a ♡ should be done at home.
Exercises marked by a † are extra exercises.

[2]This was an extra exercise in the fourth assignment.

Note : In many textbooks, you will see the notation $\frac{\partial}{\partial x^i}|_p \in T_pM$ *to mean* $[\gamma_i]$.

(d) For any $p \in M$ let

$$\mathcal{D}_p := \{X : C^\infty(M) \to \mathbb{R} \,|\, X \text{ satisfies (1)}\}$$

be the space of operators satisfying the Leibniz rule at p. We have shown that

$$\Phi : T_pM \to \mathcal{D}_p,\, Y \mapsto (f \mapsto Y(f))$$

is well defined. Show that $\mathcal{D}_p$ is a vector space and that Φ is injective.

(e) The rest of the exercise aims to prove that Φ is an isomorphism of vector spaces.

i. Let $g \in C^\infty(\mathbb{R}^n)$. Prove that there exist functions g_i such that

$$g = g(0) + \sum_{i=1}^{n} x^i g_i.$$

(Hint : Write $g(x) = g(0) + \int_0^1 \frac{d}{dt} g(tx)\, dt$.) What is $\varphi_i(0)$?

ii. Let $X \in \mathcal{D}_p$ and $f \in C^\infty(M)$. By composing with a translation if neccessary, we may assume that $\varphi(p) = 0 \in \mathbb{R}^n$. Show that there exist local functions $f_i : U \to \mathbb{R}$ such that

$$f|_U = f(p) + \sum_{i=1}^{n} (x^i \circ \varphi) f_i.$$

(You may need to reduce $\varphi(U)$ so that it is convex.)

iii. Prove that $X(f)$ only depends on the behaviour of f near p. To see this, show that for any open set V of M satisfying $p \in V \subset \overline{V} \subset U$ there exist functions $\rho, \sigma \in C^\infty(M)$ such that

$$\sigma = 1 \text{ on } V \text{ and } \operatorname{supp}(\sigma) \subset U,$$
$$\rho = 1 \text{ on } U.$$

Then consider $X((1-\rho)\sigma)$ and prove that $X(f) = X(\rho f)$. Since the choice of ρ is arbitrary and so is the size of U, we conclude that $X(f) = X(f|_U)$.

iv. Show that $X \in \Phi(T_pM)$.

2. $\diamondsuit$ Let $f : S^3 \to \mathbb{R}$ be the function $f(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2$. Show that $\frac{1}{2}$ is a regular value of f. Deduce that $f^{-1}\left(\frac{1}{2}\right)$ is a submanifold of S^3 and show that this submanifold is diffeomorphic to the torus $S^1 \times S^1$.

3. $\diamondsuit$ Consider the curve $\gamma : \mathbb{R} \to S^1 \times S^1 \; t \mapsto (e^{i2\pi t}, e^{it})$. Prove that γ is an injective immersion but that it is not an embedding. Hence $\gamma(\mathbb{R})$ is not a submanifold of the torus.

4. $\heartsuit$ Prove that $\mathrm{SU}(n)$ is a smooth manifold. Compute its dimension and its Lie algebra (i.e. compute $T_{\mathrm{id}}\,\mathrm{SU}(n)$).

5. $\heartsuit$ Prove that the set of all 2×2 matrices of rank 1 is a three-dimensional submanifold of $\mathbb{R}^4 = M(2)$. [Hint: Use the determinant function det: $M(2)\backslash\{0\} \to \mathbb{R}$]

6. † (*Smooth Urysohn Theorem:*) If A and B are disjoint, closed subsets of a smooth manifold X, Prove that there is a smooth function f on X, such that $0 \leq f \leq 1$ with $f = 0$ on A and $f = 1$ on B. [Hint: partition of unity.]

7. † (*Morse function on a manifold:*) Suppose that a point $x \in \mathbb{R}^k$, is a nondegenerate critical point of a function $f : \mathbb{R}^k \to \mathbb{R}$. We define the matrix

$$(h_{ij}) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right)$$

to be the Hessian of f at x. If the Hessian is non-singular at the critical point a, one says a is a nondegerate critical point of f. The concept of nongeneracy makes sense on manifolds, via local parametrizations. Suppose that $f : X \to \mathbb{R}$ has a critical point at $x \in X$ and that φ is a local parametrization carrying the origin to x. Then 0 is a critical point for the function $f \circ \varphi$, for $d_0(f \circ \varphi) = d_x f \circ d\varphi_0$. We shall declare x to be nondegerate for f if 0 is nondegenerate for $f \circ \varphi$. The difficulty with such local definitions is that one must always prove the cloice of parametrization to be unimportant. In this case, if φ_1 and φ_2 are two choices, then $f \circ \varphi_1 = (f \circ \varphi_2) \circ \psi$, where $\psi = \varphi_2^{-1} \circ \varphi_1$. Now prove that:

   (a) Suppose that f is a function on $\mathbb{R}^k$ with a nondegenerate critical point at 0 , and ψ is a diffeomorphism with $\psi(0) = 0$. Then $f \circ \psi$ is also a nondegenerate critical point at 0 . Observe that this result makes the nondegenerate points on a manifold well-defined. A function is Morse if all the critical points are nondegerate.

   (b) Suppose that $f = \sum_{i,j} a_{ij} x_i x_j$ in $\mathbb{R}^k$. Check that its Hessian matrix is $H = (a_{ij})$. Considering $\mathbb{R}^k$ as the vector space of column vectors, H

operates as a linear map by left multiplication, as usual. Show that if $Hv = 0$, then f is critical all along the line through v and 0 . Thus the origin is an isolated critical point iff H is non-singular.

(c) Show that the height function $h : (x_1, x_2, x_3, x_4) \mapsto x_4$ on the sphere S^3 is a Morse function with two critical points, the poles. Note that one pole is a maximum and the other a minimum.