Differential Geometry I

Lecture notes

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This is a draft. If you spot a mistake, please let me know.

Contents

Chapter 1

Smooth surfaces

1.1 The notion of a smooth surface

Let $\mathrm{U} \subset \mathbb{R}^n$ be an open subset and $f \in C^1(\mathrm{U})$. It is known from analysis that $x_0 \in \mathrm{U}$ is a point of extremum for f if

$$\frac{\partial f}{\partial x_i}(x_0) = 0$$

holds for all $i = 1, \dots, n$. Notice that this is a necessary condition, which is not sufficient in general.

A more general type of problems does not fit into this scheme. For example, consider the following.

Problem. Among all rectangular parallelepipeds, whose diagonal has a fixed length, say 1, find the one with maximal volume.

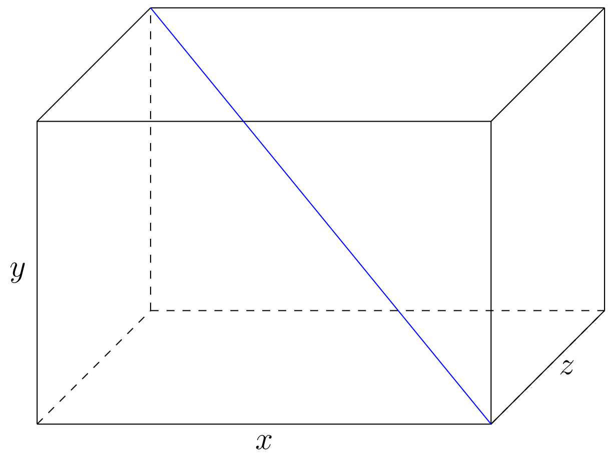


Figure 1.1: A parallelepiped

Thus, we want to find a point of maximum of the function $f(x, y, z) = xyz$ on the set

$$\mathrm{V} = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0, y > 0, z > 0 \quad \text{and} \quad x^2 + y^2 + z^2 = 1\} \subset S^2. \tag{1.1}$$

However, V is *not* an open subset of $\mathbb{R}^3$ so that the receipy known from the analysis course is not readily applicable.

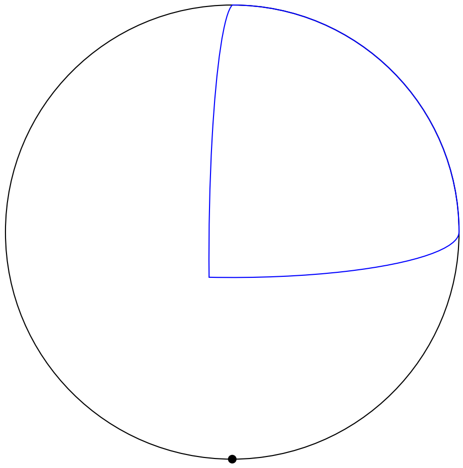


Figure 1.2: The spherical triangle $x, y, z > 0$

This problem is relatively easy to solve, however. Indeed, since $z > 0$, we obtain $z = \sqrt{1 - x^2 - y^2}$ so that we are essentially interested in the function

$$F(x, y) := f\big(x, y, \sqrt{1 - x^2 - y^2}\big) = xy\sqrt{1 - x^2 - y^2}.$$

More precisely, we want to find points of maximum of F on the set $\{(x, y) \mid x^2 + y^2 < 1, x > 0, y > 0\}$, which is an open subset of $\mathbb{R}^2$.

We compute

$$\begin{aligned} \frac{\partial F}{\partial x} &= y\sqrt{1 - x^2 - y^2} - xy\frac{x}{\sqrt{1 - x^2 - y^2}} = 0, \\ \frac{\partial F}{\partial y} &= x\sqrt{1 - x^2 - y^2} - xy\frac{y}{\sqrt{1 - x^2 - y^2}} = 0. \end{aligned} \tag{1.2}$$

Since $x \neq 0$ and $y \neq 0$, we have

$$\begin{aligned} (1.2) \quad &\iff \quad \begin{matrix} 1 - x^2 - y^2 = x^2 \\ 1 - x^2 - y^2 = y^2 \end{matrix} \quad \implies \quad x^2 = y^2 \quad \implies \quad x = y \\ &\implies \quad 3x^2 = 1 \quad \implies \quad x = y = \frac{1}{\sqrt{3}} \\ &\implies \quad z = \frac{1}{\sqrt{3}}. \end{aligned}$$

Hence, if there is a parallelepiped maximizing the volume among all rectangular parallelepipeds with the given length of the diagonal, this must be the cube.

Exercise 1.3. Show that $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is a point of maximum indeed.

Consider a more general problem of constrained maximum/minimum. Given $f, \varphi \in C^\infty(\mathbb{R}^n)$ find a point of maximum/minimum of f on the set

$$S := \{x \in \mathbb{R}^n \quad | \quad \varphi(x) = 0\}.$$

Proposition 1.4. *Assume that for $p \in S$ we have*

$$\frac{\partial \varphi}{\partial x_n}(p) \neq 0. \tag{1.5}$$

Then there is a neighbourhood W of p in $\mathbb{R}^n$, an open subset $\mathrm{V} \subset \mathbb{R}^{n-1}$, and a smooth function $\psi \colon \mathrm{V} \to \mathbb{R}$ such that for $x = (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}$ we have

$$x \in S \cap \mathrm{W} \qquad \Longleftrightarrow \qquad y \in \mathrm{V} \quad \text{and} \quad z = \psi(y).$$

This is a celebrated implicit function theorem, whose proof was given in the analysis course.

Theorem 1.6. *Let $p \in S$ be a point of (local) maximum of f on S. If (1.5) holds, then there exists some $\lambda \in \mathbb{R}$ such that*

$$\frac{\partial f}{\partial x_j}(p) = \lambda \frac{\partial \varphi}{\partial x_j}(p) \qquad \Longleftrightarrow \qquad \nabla f(p) = \lambda \nabla \varphi(p) \tag{1.7}$$

holds for each $j = 1, \dots, n$.

Proof. Let $p = (y_0, z_0)$ be a local maximum for f on S. Hence, y_0 is a local maximum for the function

$$F \colon \mathrm{V} \to \mathbb{R}, \qquad F(y) := f(y, \psi(y))$$

This yields

$$\frac{\partial F}{\partial y_j}(y_0) = \frac{\partial f}{\partial y_j}(p) + \frac{\partial f}{\partial x_n}(p)\,\frac{\partial \psi}{\partial y_j}(y_0) = 0$$

for all $j \leq n-1$.

Furthermore, since $\varphi(y, \psi(y)) \equiv 0$, we have

$$\frac{\partial \varphi}{\partial y_j} + \frac{\partial \varphi}{\partial x_n}\frac{\partial \psi}{\partial y_j} \equiv 0.$$

This yields in turn

$$\frac{\partial \psi}{\partial y_j}(y_0) = -\frac{\partial \varphi}{\partial y_j}(p) \Big/ \frac{\partial \varphi}{\partial x_n}(p) \qquad \Longrightarrow \qquad \frac{\partial f}{\partial y_j}(p) = \left(\frac{\partial f}{\partial x_n}(p) \Big/ \frac{\partial \varphi}{\partial x_n}(p)\right) \cdot \frac{\partial \varphi}{\partial y_j}(p).$$

Thus, (1.7) holds for all $j \leq n-1$ with $\lambda := \frac{\partial f}{\partial x_n}(p) / \frac{\partial \varphi}{\partial x_n}(p)$ independent of j.

For $j = n$ we have

$$\frac{\partial f}{\partial x_n}(p) = \left(\frac{\partial f}{\partial x_n}(p) \Big/ \frac{\partial \varphi}{\partial x_n}(p)\right) \cdot \frac{\partial \varphi}{\partial x_n}(p) = \lambda \frac{\partial \varphi}{\partial x_n}(p).$$

Thus, (1.7) holds also for $j = n$ with the same λ. $\square$

Let us come back to the example about maximal value of parallelepipeds with a fixed length of the diagonal. Thus, if (x, y, z) is a point of maximum of f on (1.1), then there exists $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} yz &= 2\lambda x \\ xz &= 2\lambda y \\ xy &= 2\lambda z \end{aligned} \qquad \Longrightarrow \qquad (xyz)^2 = 8\lambda^3 xyz \qquad \Longrightarrow \qquad xyz = 8\lambda^3.$$

This yields in turn

$$8\lambda^3 = xyz = x(yz) = 2\lambda x^2.$$

Notice that $\lambda \neq 0$, since otherwise $x = 0$ or $y = 0$ or $z = 0$. Hence, we obtain $x = 2\lambda$.

A similar argument yields also $y = 2\lambda$ and $z = 2\lambda$. Therefore we obtain

$$4\lambda^2 + 4\lambda^2 + 4\lambda^2 = 1 \qquad \Longrightarrow \qquad \lambda = \frac{1}{2\sqrt{3}} \qquad \Longrightarrow \qquad x = y = z = \frac{1}{\sqrt{3}},$$

which is in agreement with our previous computation.

Coming back to Proposition 1.4, it is clear that it is only important that one of the partial derivatives of φ does not vanish. This leads to the following definition.

Definition 1.8 (Surface)**.** A non-empty set $S \subset \mathbb{R}^3$ is called a (smooth) *surface*, if for any $p \in S$ there exists an open set $\mathrm{V} \subset \mathbb{R}^2$ and a smooth map $\psi : \mathrm{V} \to \mathbb{R}^3$ such that the following holds:

(i) $\psi(\mathrm{V}) =: \mathrm{U}$ is a neighbourhood of p in S; in particular, $\psi(\mathrm{V}) \subset S$.

(ii) $\psi \colon \mathrm{V} \to \mathrm{U}$ is a homeomorphism.

(iii) $D_q\psi \colon \mathbb{R}^2 \to \mathbb{R}^3$ is injective $\forall q \in \mathrm{V}$.

Example 1.9. Assume $\varphi \in C^\infty(\mathbb{R}^3)$ satisfies

$$\frac{\partial \varphi}{\partial z}(p) \neq 0 \qquad \text{for all} \quad p \in S := \varphi^{-1}(0).$$

Let ψ be as in Proposition 1.28. Define $\Psi(x, y) := (x, y, \psi(x, y))$. If U and V are also as in Proposition 1.28, then $\Psi \colon \mathrm{V} \to S \cap \mathrm{U}$ is a homeomorphism, since $\pi \colon S \cap \mathrm{U} \to \mathrm{V}, \pi(x, y, z) = (x, y)$ is a continuous inverse. Furthermore,

$$D\Psi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_x\psi & \partial_y\psi \end{pmatrix}$$

is clearly injective at all points. Hence, S is a surface.

Again, the same conclusion holds if we assume only that $\nabla\varphi(p) \neq 0$ for all $p \in \varphi^{-1}(0)$. In particular,

- the sphere $S^2 = \{x^2 + y^2 + z^2 = 1\}$
- the cylinder $C = \{(x, y, z) \mid x^2 + y^2 = 1\}$
- the hyperboloid $H = \{x^2 + y^2 - z^2 = 1\}$

are surfaces

Example 1.10 (Torus)**.** Let C be the circle of radius r in the yz-plane centered at the point $(0, a, 0)$ as shown on Fig. 1.4, where $a > r$.

More formally,

$$T := \left\{\left(\sqrt{x^2 + y^2} - a\right)^2 + z^2 = r^2\right\}.$$

Exercise 1.11. Check that T is a surface indeed.

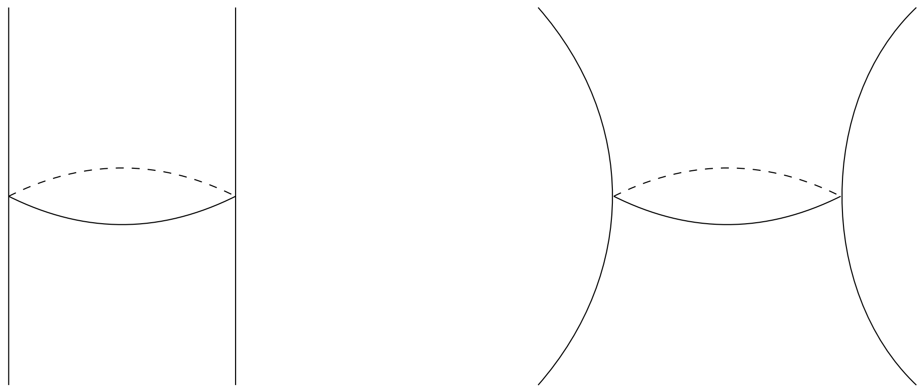


Figure 1.3: The cylinder and hyperboloid

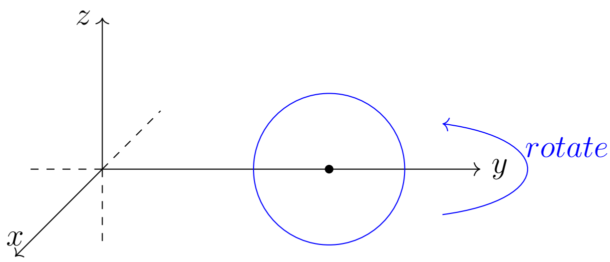


Figure 1.4: The torus as a circle rotated with respect to an axis

Example 1.12 (A non-example)**.** The double cone $C_0 := \{x^2 + y^2 - z^2 = 0\}$ is not a surface. Indeed, assume C_0 is a surface. Then the tip of the cone p must have a neighbourhood U homeomorphic to an open disc in $\mathbb{R}^2$.

Let $f\colon \mathrm{U} \to D$ be a homeomorphism. Then $f\colon \mathrm{U}\backslash\{p\} \to D\backslash\{f(p)\}$ is also a homeomorphism. However, this is impossible, since the punctured disc is connected but $\mathrm{U}\backslash\{p\}$ is disconnected. Hence, p does not have a neighbourhood homeomorphic to a disc (or any open subset of $\mathbb{R}^2$).

Exercise 1.13. Show that a straight line is not a surface.

Remark 1.14.

1) The map ψ in the definition of the surface is called a *parametrization*.
2) Condition (iii) is equivalent to the following:

$$\partial_u\psi \quad \text{and} \quad \partial_v\psi \quad \text{are linearly independent}$$

at each point $(u, v) \in \mathrm{V}$.

Proposition 1.15. *Let S be a surface. For any $p \in S$ there exists a neighbourhood $W \subset \mathbb{R}^3$ and $\varphi \in C^\infty(W)$ such that*

$$S \cap W = \{x \in W \mid \varphi(x) = 0\} \quad \textit{and} \quad \nabla\varphi(x) \neq 0$$

for any $x \in S \cap W$.

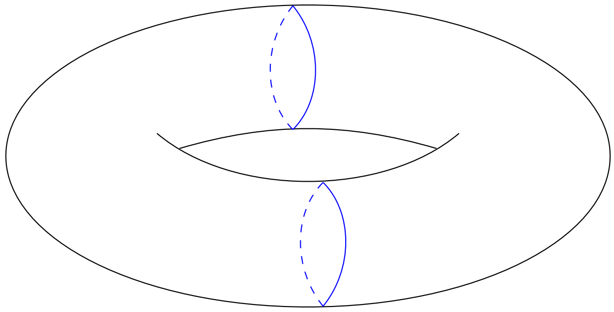


Figure 1.5: The torus

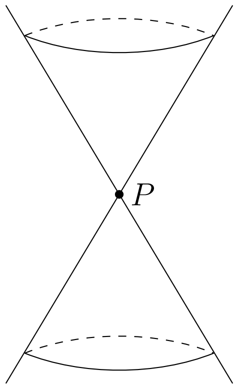


Figure 1.6: The double cone

Proof. Choose a parametrization $\psi\colon \mathrm{V} \to \mathrm{U} \subset S$. Let $(u_0, v_0) \in \mathrm{V}$ be a unique point such that $\psi(u_0, v_0) = p$. Choose a vector $n \in \mathbb{R}^3$ such that

$$\partial_u \psi(u_0, v_0), \quad \partial_v \psi(u_0, v_0), \quad n \tag{1.16}$$

are linearly independent. Consider the map

$$\Psi\colon \mathrm{V} \times \mathbb{R} \to \mathbb{R}^3, \qquad \Psi(u, v, w) = \psi(u, v) + w \cdot n$$

The linear independence of (1.16) yields $\det D\Psi(u_0, v_0, 0) \neq 0$. By the inverse map theorem, there exists an open neighbourhood $W \subset \mathbb{R}^3$ of p and a smooth map $\Phi\colon W \to V \times \mathbb{R} \subset \mathbb{R}^3$ such that

$$\Psi \circ \Phi(x) = x \qquad \forall\, x \in W.$$

If $\Phi = (\varphi_1, \varphi_2, \varphi_3)$, then

$$\Psi \circ \Phi(x) = \psi(\varphi_1(x), \varphi_2(x)) + \varphi_3(x) \cdot n = x.$$

Observe that

$$x \in S \cap W \qquad \Longleftrightarrow \qquad \exists\, (u, v) \in \mathrm{V} \textit{ such that } \psi(u, v) = x$$

and consequently

$$\Psi(u, v, 0) = \psi(u, v) = x = \Psi(\varphi_1(x), \varphi_2(x), \varphi_3(x)).$$

Since Ψ is injective (on an open neighbourhood of $(u_0, v_0, 0)$), we have

$$x \in S \cap W \qquad \Longleftrightarrow \qquad \varphi_3(x) = 0.$$

Furthermore, since $\det D\Phi(x) \neq 0$ for all $x \in W$, the vectors $\nabla\varphi_1(x), \nabla\varphi_2(x), \nabla\varphi_3(x)$ are linearly independent at each $x \in W$. In particular, $\nabla\varphi_3(x) \neq 0$ for all $x \in W$. □

The following corollary follows immediately from Proposition 1.15.

Corollary 1.17. *Any surface is locally the graph of a smooth function.* □

Example 1.18 (A non-example)**.** The union of two intersecting planes in $\mathbb{R}^3$ is *not* a surface. Indeed, assume that

$$S := \{z = 0\} \cup \{x = 0\}$$

is a surface. Then there exists a smooth function φ defined in a neighbourhood W of the origin such that φ vanishes on S and $\nabla\varphi(0) \neq 0$ by Proposition 1.15. Notice that φ vanishes identically along S, hence φ vanishes identically along all three coordinate axes (at least in a neighbourhood of the origin). This yields in turn $\nabla\varphi(0) = 0$, which is a contradiction.

Exercise 1.19. Show that the cone $C := \{x^2 + y^2 - z^2 = 0,\ z \geq 0\}$ is not a smooth surface, cf. Example 1.12 above.

1.2 The change of coordinates maps

Neither parametrizations, nor local functions as in the Proposition 1.15 are unique. Our next goal is to understand a relation between different parametrizations.

Thus, let

$$\psi_1 \colon \mathrm{V}_1 \longrightarrow \mathrm{U}_1 \subset S \qquad \text{and} \qquad \psi_2 \ : \mathrm{V}_2 \longrightarrow \mathrm{U}_2 \subset S$$

be two parametrizations such that $\mathrm{U}_1 \cap \mathrm{U}_2 \neq 0$. Since both ψ_1 and ψ_2 are homeomorphisms, we have a well-defined continuous map

$$\psi_{21} := \psi_2^{-1} \circ \psi_1 \colon \mathrm{V}_{12} \longrightarrow \mathrm{V}_{21}$$

which is called "a transition map" or "a change of coordinates map".

Notice that ψ_{21} is a map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined on an open subset. Therefore, transition maps can be studied by the tools familiar from the analysis course.

Example 1.20. Consider the sphere S^2, which can be covered by the images of two parametrizations as follows. The inverse of the steregraphic projection from the north pole N is given by

$$(u, v) \longmapsto \psi_N(u, v) = \frac{1}{1 + u^2 + v^2}\left(2u, 2v, -1 + u^2 + v^2\right)$$

This is a homeomorphism viewed as a map $\mathbb{R}^2 \longrightarrow S^2 \backslash \{N\}$ and is clearly smooth.

Exercise 1.21. Show that $D\psi_N$ is injective at each point.

Thus, ψ_N is a parametrization (at each point $p \in S^2 \backslash \{N\}$). Of course, we have also the inverse ψ_S of the stereographic projection from the south pole S. The images of these two parametrizations cover together the whole sphere S^2. A straightforward computation shows that the change of coordinates map $\psi_{SN} := \psi_S^{-1} \circ \psi_N \colon \mathbb{R}^2 \backslash \{0\} \longrightarrow \mathbb{R}^2 \backslash \{0\}$ is given by

$$\psi_{SN}(u, v) = \frac{1}{u^2 + v^2}(u, v)$$

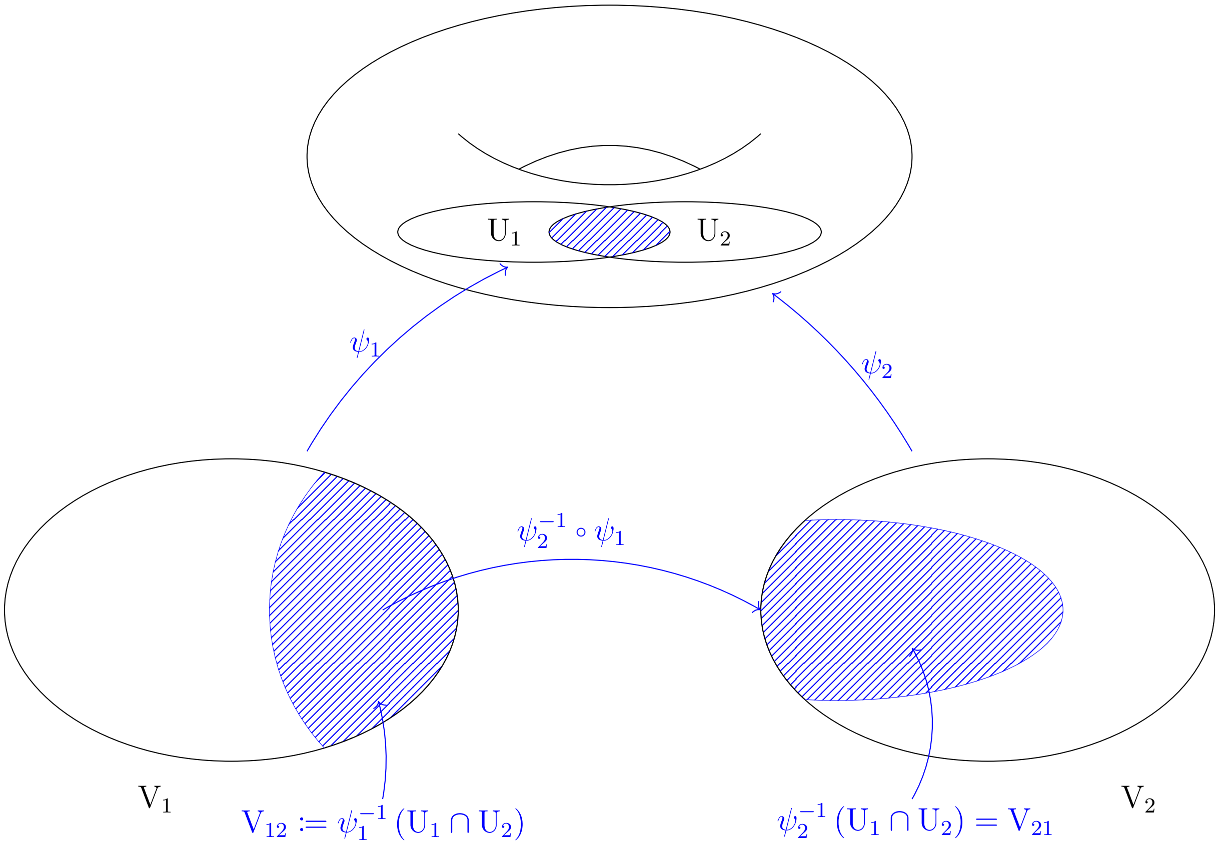


Figure 1.7: The transition map

Exercise 1.22. Show that the sphere can not be covered by the image of a single parametrization.

Theorem 1.23. *Let S be a surface. For any two parametrizations ψ_1 and ψ_2 as above, the change of coordinates map ψ_{12} is smooth.*

Proof. Since smoothness is a local property, it suffices to show that for all $(u_0, v_0) \in V_{12}$ there exists a neighbourhood $V_0 \subset V_{12}$ such that $\psi_{21}\big|_{V_0}$ is smooth.

Thus, set $p_0 := \psi_1(u_0, v_0)$. For this p_0 and ψ_2 construct a smooth map $\Phi_2 \colon W \longrightarrow V_2 \times \mathbb{R}$ as in the proof of the Proposition 1.15. Recall that

$$\Phi_2\big|_{S\cap W} \colon S \cap W \longrightarrow V_2 \times \{0\} = V_2$$

equals ψ_2^{-1}.

The map $\Phi_2 \circ \psi_1 \colon \psi_1^{-1}(S \cap W) \to V_2$ is clearly smooth as a composition of smooth maps. Set $V_0 := V_{12} \cap \psi_1^{-1}(S \cap W)$. Since the image of ψ_1 lies in S, we obtain that

$$\Phi_2 \circ \psi_1\big|_{V_0} = \psi_2^{-1} \circ \psi_1\big|_{V_0} = \psi_{21}\big|_{V_0}$$

is smooth. □

1.3 Smooth functions on surfaces

Definition 1.24. Let S be a surface. A function $f \colon S \to \mathbb{R}$ is said to be smooth, if for any parametrization $\psi \colon V \to U$ the composition

$$F := f \circ \psi \colon V \longrightarrow \mathbb{R}$$

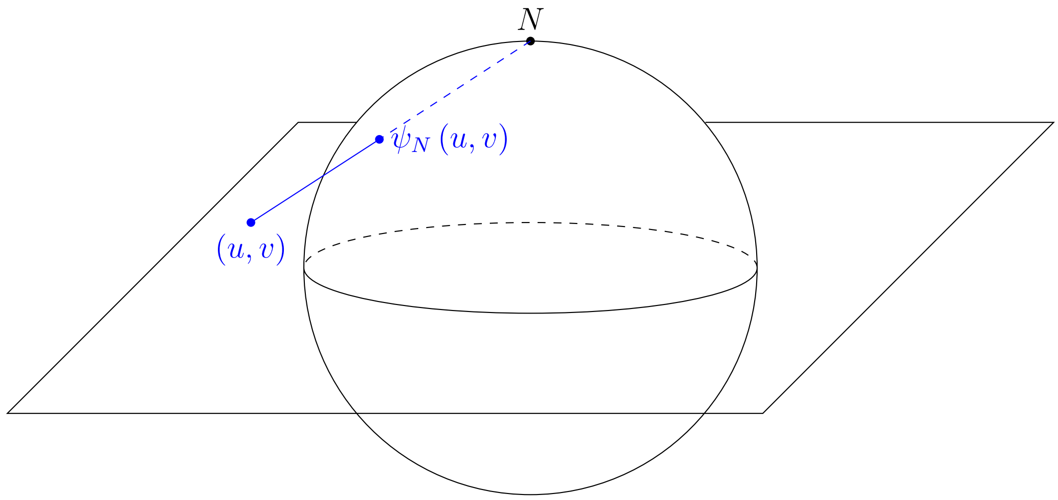


Figure 1.8: The inverse of the stereographic projection

is smooth. The function $F := f \circ \psi$ is called a local (coordinate) representation of f.

Remark 1.25. Theorem 1.23 imples that if $f \circ \psi_1$ is smooth, then $f \circ \psi_2$ is also smooth on $\mathrm{V}_{21} = \psi_2^{-1}(\mathrm{U}_1 \cap \mathrm{U}_2)$. Indeed,

$$f \circ \psi_2 = f \circ \psi_1 \circ \left(\psi_1^{-1} \circ \psi_2\right) = (f \circ \psi_1) \circ \psi_{12}$$

$f \circ \psi_1$ and ψ_{12} are smooth. Hence, if (V_i, ψ_i) is a collection of parametrizations such that $\psi_i(\mathrm{V}_i)$ covers all of S, it suffices to check that $f \circ \psi_i$ is smooth for all i.

Example 1.26. Let $h\colon \mathbb{R}^3 \to \mathbb{R}$ be an arbitrary smooth function. Define $f\colon S \to \mathbb{R}$ as the restriction of h. Then f is smooth, since for any parametrization ψ we have $f \circ \psi = h \circ \psi$ and the right hand side is clearly smooth.

For example, for any fixed $a \in \mathbb{R}^3$ the height function

$$f_a(x) = \langle a, x\rangle \qquad x \in S$$

is a smooth function on S. In particular, set $S = S^2$ and $h(x,y,z) = z$. Then the coordinate representation of $f = h\big|_{S^2}$ with respect to ψ_N is

$$F(u,v) = f \circ \psi_N(u,v) = \frac{-1+u^2+v^2}{1+u^2+v^2}.$$

This can be seen as a sanity check: This function is smooth indeed.

Example 1.27. Let $\psi\colon \mathrm{V} \to \mathrm{U}$ be a parametrization of a surface S. Since ψ is a homeomorphism, we have the inverse map

$$\varphi := \psi^{-1}\colon \mathrm{U} \longrightarrow \mathrm{V}.$$

Since U itself is a surface (with a single parametrization ψ), it makes sense to ask if φ viewed as a map $\mathrm{U} \to \mathbb{R}^2$ is smooth, which means by definition that both components of φ are smooth functions. This is the case indeed, since the local representation of φ is nothing else but $\varphi \circ \psi = \mathrm{id}$, which is surely smooth. Any such pair (U, φ) is called *a chart* on S.

Proposition 1.28. *Let S be a surface. Then the set $C^\infty(S)$ of all smooth functions on S is a vector space, that is*

$$\begin{matrix} f, g \in C^\infty(S) \\ \lambda, \mu \in \mathbb{R} \end{matrix} \qquad \Longrightarrow \qquad \lambda f + \mu g \in C^\infty(S).$$

In fact, we also have

$$f, g \in C^\infty(S) \qquad \Longrightarrow \qquad f \cdot g \in C^\infty(S),$$

where $f \cdot g$ is the product-function $p \mapsto f(p) \cdot g(p)$.

Proof. We prove the last statement only, while the first one is left as an exercise to the reader. If $\psi\colon \mathrm{U} \to \mathrm{V}$ is a parametrization, then $(f \cdot g) \circ \psi = (f \circ \psi) \cdot (g \circ \psi)$. Since $(f \circ \psi) \in C^\infty(\mathrm{V})$ and $(g \circ \psi) \in C^\infty(\mathrm{V})$, the function $(f \cdot g) \circ \psi$ is smooth as the product of smooth functions of two variables. $\square$

Let $W \subset \mathbb{R}^n$ be an open set.

Definition 1.29. A continuous map $f\colon W \longrightarrow S$, where S is a surface, is called *smooth*, if for any parametrization $\psi\colon \mathrm{V} \to \mathrm{U} \subset S$ the map

$$\varphi \circ f = \psi^{-1} \circ f\colon f^{-1}(\mathrm{U}) \longrightarrow \mathrm{V} \subset \mathbb{R}^2$$

is smooth.

In the above definition we require that f is continuous to ensure that $f^{-1}(\mathrm{U})$ is an open subset so that it makes sense to talk about smoothness of the coordinate representation $\varphi \circ f$.

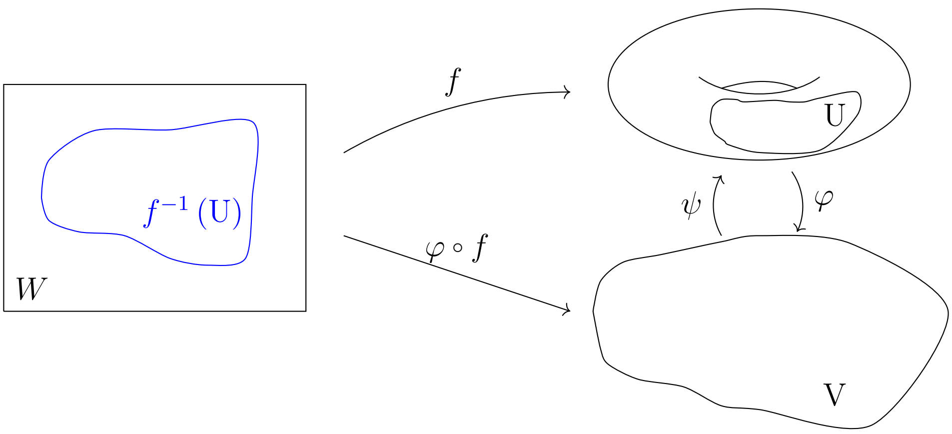


Figure 1.9: A map into a surface and its coordinate representation

Proposition 1.30. *$f\colon W \to S$ is smooth if and only if f is smooth as a map $W \to \mathbb{R}^3$. More formally, this means the following: If $\iota\colon S \to \mathbb{R}^3$ denotes the natural inclusion map, then*

$$f \in C^\infty(W; S) \qquad \Longleftrightarrow \qquad \iota \circ f \in C^\infty\left(W; \mathbb{R}^3\right)$$

Proof. Pick a parametrization ψ of S and construct a smooth map $\Phi\colon X \to \mathbb{R}^3$ just as in the proof of Proposition 1.15, where $X \subset \mathbb{R}^3$ is an open set. Assume $f\colon W \to \mathbb{R}^3$ is smooth. Then $\Phi \circ f$ is also smooth as the composition of smooth maps. However, since f takes values in S and $\Phi|_S = \varphi = \psi^{-1}$, we obtain that $\varphi \circ f = \Phi \circ f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is smooth.

Conversely, assume that $f\colon W \to S$ is smooth. Then

$$f\big|_{f^{-1}(\mathrm{U})} = (\psi \circ \varphi) \circ f\big|_{f^{-1}(\mathrm{U})} = \psi \circ (\varphi \circ f)\big|_{f^{-1}(\mathrm{U})}$$

is again smooth as the composition of smooth maps. $\square$

The following class of maps will be particularly important in the sequel.

Definition 1.31. Let $I \subset \mathbb{R}$ be an (open) interval. A smooth map $\gamma\colon I \to S$ is called a smooth curve on S.

If $0 \in I$, we say that γ is a smooth curve through $p := \gamma(0) \in S$.

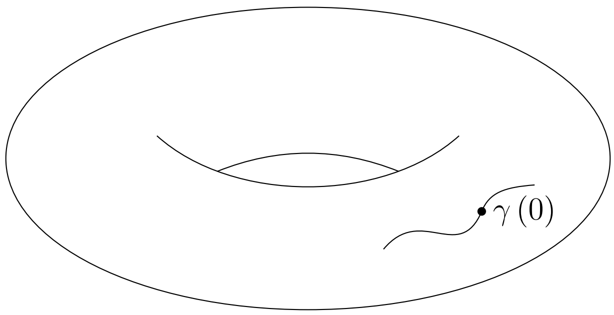


Figure 1.10: A smooth curve on a surface

Example 1.32. Let $p \in S^2$ and $v \in \mathbb{R}^3$ such that $\langle p, v\rangle = 0$ and $\|v\| = 1$. Define $\gamma_v\colon \mathbb{R} \to \mathbb{R}^3$ by $\gamma_v(t) = (\cos t) \cdot p + (\sin t) \cdot v$. Since

$$\begin{aligned}\|\gamma_v(t)\|^2 &= \langle \cos t \cdot p + \sin t\, v, \cos t\, p + \sin t \cdot v\rangle \\ &= \cos^2 t \cdot \|p\|^2 + 0 + \sin^2 t \cdot \|v\|^2 \\ &= \cos^2 t + \sin^2 t = 1,\end{aligned}$$

we obtain that $\gamma_v\colon \mathbb{R} \to S^2$ is a smooth curve through p. Of course, the image of γ_v is a great circle on S^2.

Even more generally, we can define smooth maps between surfaces as follows.

Definition 1.33. Let S_1 and S_2 be two surfaces. A continuous map $f\colon S_1 \to S_2$ is said to be smooth, if for any parametrizations $\psi\colon \mathrm{V} \to \mathrm{U} \subset S_1$ and $\chi\colon W \to X \subset S_2$ the map

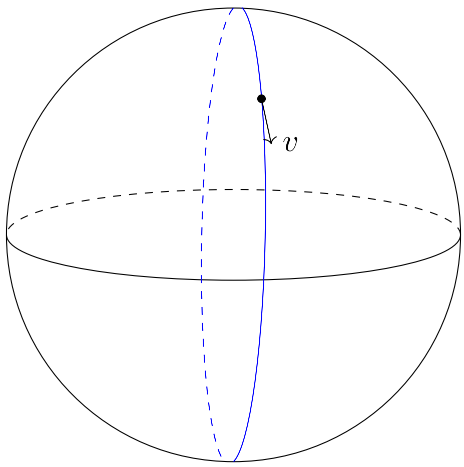
$$\chi^{-1} \circ f \circ \psi\colon \psi^{-1}\left(f^{-1}(X)\right) \longrightarrow W \tag{1.34}$$

is smooth. Just like in the case of functions, (1.34) is called the coordinate (or local) representation of f.

Remark 1.35. Since parametrizations and charts contain the same amount of information, we can also define smoothness of a map $f\colon S_1 \to S_2$ in terms of charts as follows: f is smooth if and only if for any chart (U, φ) on S_1 and any chart (X, ξ) on S_2 the map

$$\xi \circ f \circ \varphi^{-1}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

is smooth (on an open subset where defined). The map $\xi \circ f \circ \varphi^{-1}$ is also called a coordinate representation of f (with respect to charts (U, φ) and (X, ξ)).



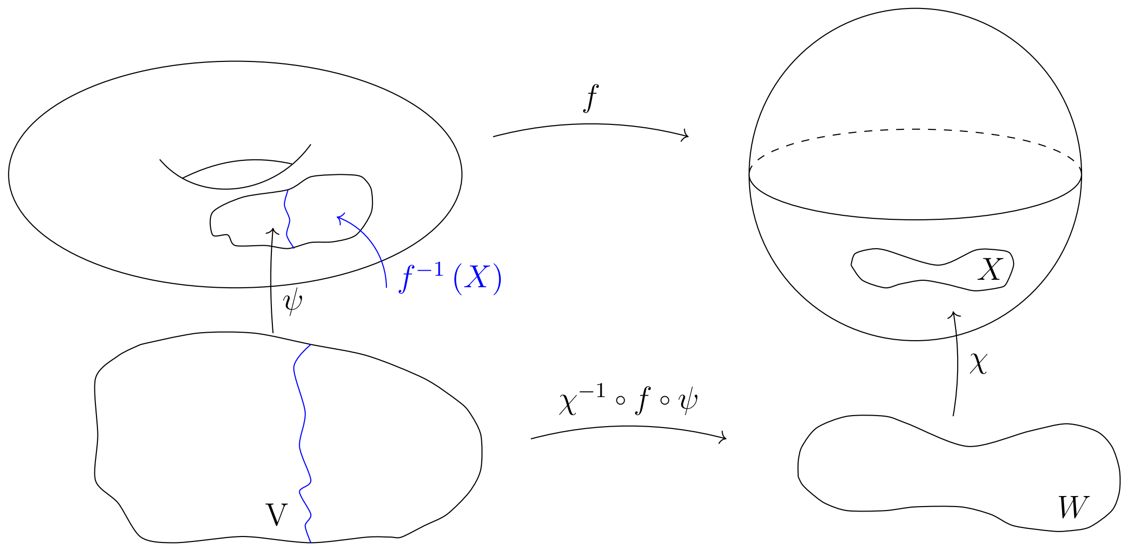


Figure 1.11: A smooth map between surfaces and its coordinate representation

Remark 1.36. Just like in the case of functions, it suffices to find two collections $\{\psi_i \colon \mathrm{V}_i \to \mathrm{U}_i\}$ and $\{\chi_j \colon W_j \to X_j\}$ of parametrizations such that

$$\bigcup_i \mathrm{U}_i = S_1 \qquad \textit{and} \qquad \bigcup_j X_j = S_2$$

and check that all coordinate representations $\chi_j^{-1} \circ f \circ \psi_i$ are smooth.

Consider the antipodal map

$$a \colon S^2 \to S^2, \quad a(x) = -x.$$

For any $(u, v) \in \mathbb{R}^2$ we have

$$a \circ \psi_N(u, v) = -\frac{1}{1 + u^2 + v^2}\left(2u, 2v, -1 + u^2 + v^2\right)$$

Since $\psi_S^{-1} \colon S^2 \backslash \{S\} \to \mathbb{R}^2$ is given by

$$(x, y, z) \longmapsto \left(\frac{x}{1+z}, \frac{y}{1+z}\right),$$

we obtain

$$\begin{aligned}\psi_S^{-1} \circ a \circ \psi_N (u, v) &= \frac{1}{1 + \frac{1-u^2-v^2}{1+u^2+v^2}} \left(-\frac{2u}{1+u^2+v^2}, -\frac{2v}{1+u^2+v^2} \right) \\ &= -\frac{1+u^2+v^2}{2} \left(\frac{2u}{1+u^2+v^2}, \frac{-2v}{1+u^2+v^2} \right) \\ &= -(u, v)\end{aligned}$$

It follows in a similar manner, that $\psi_S^{-1} \circ a \circ \psi_S$, $\psi_N^{-1} \circ a \circ \psi_N$, and $\psi_N^{-1} \circ a \circ \psi_S$ are also smooth. Hence, a is smooth.

Proposition 1.37. *Let $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth map such that $h(S_1) \subset S_2$, where S_1 and S_2 are surfaces. Then $h\big|_{S_1} : S_1 \to S_2$ is also smooth.*

The proof of this proposition is similar to the proof of Proposition 1.30 and is left as an exercise to the reader.

To construct a more interesting example, pick a polynomial

$$p(z) \coloneqq z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

with complex coefficients. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can view p as a smooth map $\mathbb{R}^2 \to \mathbb{R}^2$. Define $f\colon S^2 \to S^2$ by

$$f(p) = \begin{cases} \psi_N \circ p \circ \psi_N^{-1}(p) & \text{if } p \neq N, \\ N & \text{if } p = N. \end{cases} \tag{1.38}$$

I claim that f is smooth. Indeed, since by the construction of f, the coordinate representation of f with respect to the pair $(\mathbb{R}^2, \psi_N)$ and $(\mathbb{R}^2, \psi_N)$ of parametrizations (the first one on the source of f, the second one on the target), is

$$\psi_N^{-1} \circ f \circ \psi_N = \underbrace{\psi_N^{-1} \circ \psi_N}_{\text{id}} \circ p \circ \underbrace{\psi_N^{-1} \circ \psi_N}_{\text{id}} = p.$$

Hence f is smooth at each point $p \in S^2 \setminus \{N\}$. To check that f is also smooth at N too, consider

$$\psi_S \circ f \circ \psi_S^{-1}(z) = \begin{cases} \psi_S \circ \psi_N^{-1} \circ p \circ \psi_N \circ \psi_S^{-1} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

We know that

$$\begin{aligned} \psi_{SN}(z) = \psi_S \circ \psi_N^{-1}(z) = \frac{1}{|z|^2} z = \frac{1}{z \cdot \bar{z}} \cdot z = \frac{1}{\bar{z}} \\ \implies \quad \psi_{NS}(z) = \psi_{SN}^{-1}(z) = \frac{1}{\bar{z}}\,. \end{aligned}$$

Hence, we compute

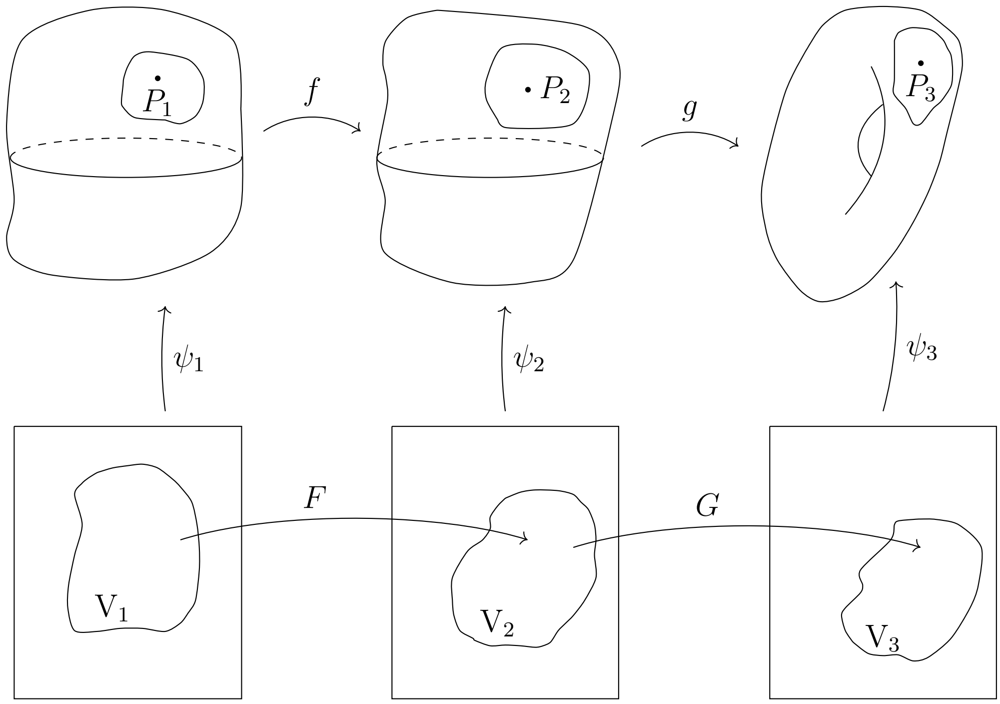
$$\begin{aligned}\psi_{SN} \circ p \circ \psi_{NS}(z) &= \psi_{SN} \left(\frac{1}{\bar{z}^n} + \frac{a_{n-1}}{\bar{z}^{n-1}} + \ldots + a_0 \right) \\ &= \psi_{SN} \left(\frac{1 + a_{n-1}\bar{z} + \ldots + a_0 \bar{z}^n}{\bar{z}^n} \right) \\ &= \frac{z^n}{1 + \bar{a}_{n-1} z + \ldots + \bar{a}_0 z^n}\,, \qquad \text{if } z \neq 0.\end{aligned}$$

This yields that $\psi_S \circ f \circ \psi_S^{-1}$ is smooth even at $z = 0$, that is f is smooth everywhere on S (or, simply, f is smooth).

Theorem 1.39. *Suppose $f\colon S_1 \to S_2$ and $g\colon S_2 \to S_3$ are smooth maps between surfaces. Then $g \circ f\colon S_1 \to S_3$ is also smooth.*

Proof. Pick a point $p_1 \in S_1$ and denote $p_2 := f(p_1) \in S_2$, $p_3 := g(p_2) = g(f(p_1)) \in S_3$. Pick parametrizations

$$\psi_j\colon \mathrm{V}_j \longrightarrow \mathrm{U}_j \subset S_j.$$



In a sufficiently small neighbourhood of p_1 we have

$$\psi_3^{-1} \circ (g \circ f) \circ \psi_1 = \underbrace{\psi_3^{-1} \circ g \circ \psi_2}_{G \in C^\infty} \circ \underbrace{\psi_2^{-1} \circ f \circ \psi_1}_{F \in C^\infty}.$$

Hence, $g \circ f$ is smooth in a neighbourhood of p_1. Since p_1 was arbitrary, $g \circ f$ is smooth everywhere. □

Remark 1.40. The proof shows that the coordinate representation of the composition is the composition of coordinate representations.

Notice that Theorem 1.39 yields in particular the following: If $\gamma\colon I \to S_1$ is a smooth curve and $f\colon S_1 \to S_2$ is a smooth map, then $f \circ \gamma\colon I \to S_2$ is also a smooth curve.

Definition 1.41. A smooth map $f\colon S_1 \to S_2$ is called a diffeomorphism, if there exists a smooth map $g\colon S_2 \to S_1$ such that

$$g \circ f = \mathrm{id}_{S_1} \qquad \textit{and} \qquad f \circ g = \mathrm{id}_{S_2}$$

Example 1.42. The antipodal map $a\colon S^2 \to S^2$ is a diffeomorphism.

Example 1.43. The hyperboloid $H = \{x^2 + y^2 - z^2 = 1\}$ and cylinder $C = \{x^2 + y^2 = 1\}$ are diffeomorphic, that is there exists a diffeomorphism $f\colon H \to C$. Explicitly, define

$$h\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \quad \textit{by} \quad h(x, y, z) = \left(\frac{x}{\sqrt{1+z^2}}, \frac{y}{\sqrt{1+z^2}}, z \right)$$

Clearly, $h \in C^\infty(\mathbb{R}^3;\mathbb{R}^3)$. If $(x,y,z) \in H$, then $\left(\frac{x}{\sqrt{1+z^2}}\right)^2 + \left(\frac{y}{\sqrt{1+z^2}}\right)^2 = \frac{x^2+y^2}{1+z^2} = 1$, that is $f := h\big|_H : H \to C$ is smooth.

Exercise 1.44. Show that the restriction of $h^{-1}\colon \mathbb{R}^3 \to \mathbb{R}^3$ given explicitly by

$$h^{-1}(u,v,w) = \left(\sqrt{1+w^2}\,u, \sqrt{1+w^2}\,v, w\right)$$

yields a smooth inverse of f.

Remark 1.45. A map $f\colon S_1 \to S_2$ may fail to be a diffeomorphism in the following two ways: either f^{-1} does not exist or f^{-1} exists but is not smooth.

Example 1.46 (A non-example)**.** Consider a map

$$f\colon C \longrightarrow C, \quad f(x,y,z) = \left(x,y,z^3\right),$$

which is smooth. The inverse $f^{-1}\colon C \to C$ exists:

$$f^{-1}(x,y,z) = \left(x,y,\sqrt[3]{z}\right).$$

It is continuous, but fails to be smooth.

Exercise 1.47. Compute a coordinate representation of f^{-1} and check that this fails to be smooth indeed.

Example 1.48. Let S be a smooth surface and let $\psi\colon \mathrm{V} \to \mathrm{U}$ be any parametrization. Consider U as a surface covered by the image of a single parametrization ψ. Then $\varphi = \psi^{-1}$ exists and is smooth as we have seen in Example 1.27. That is U is diffeomorphic to V, which is an open subset of $\mathbb{R}^2$. Summing up, we see that any surface is locally diffeomorphic to an open subset of $\mathbb{R}^2$.

Exercise 1.49.

(i) Show that the disc $D := \left\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\right\}$ is diffeomorphic to $\mathbb{R}^2$, that is there exists a smooth bijective map $f\colon D \to \mathbb{R}^2$ such that $f^{-1}\colon \mathbb{R}^2 \to D$ is also smooth.

(ii) Show that any smooth surface is locally diffeomorphic to $\mathbb{R}^2$, that is any point $p \in S$ has a neighbourhood U diffeomorphic to $\mathbb{R}^2$.

1.4 The tangent plane

Let S be a surface.

Definition 1.50. A vector $v \in \mathbb{R}^3$ is said to be tangent to S at p, if there exists a smooth curve $\gamma\colon (-\varepsilon,\varepsilon) \to S$ such that

$$\gamma(0) = p \quad \textit{and} \quad \dot\gamma(0) = v.$$

Notice that when computing the tangent vector of γ we think of γ as a curve in $\mathbb{R}^3$.

The set T_pS of all vectors tangent to S at the point p is called *the tangent space* of S at p.

Example 1.51. For $S = S^2$ and an arbitrary point p we have the curve

$$\gamma\colon \mathbb{R} \to S^2, \qquad \gamma_v(t) = \cos t \cdot p + \sin t \cdot v,$$

where $\|v\| = 1$ and $v \perp p$ just as in in Example 1.32. Then $\dot\gamma_v(0) = v$. Hence, v is tangent to S^2 at p.

In fact, any vector v which is orthogonal to p is tangent to S^2 at p. Indeed, set $\lambda := \|v\|$ and $v_1 := \lambda^{-1} v$, and

$$\gamma\colon \mathbb{R} \to S^2, \qquad \gamma(t) = \gamma_{v_1}(\lambda t).$$

Then $\gamma(0) = p$ and $\dot\gamma(0) = \lambda\dot\gamma_{v_1}(0) = v$.

Proposition 1.52. *Let $\psi\colon \mathrm{V} \to \mathrm{U}$ be a parametrization such that $\psi(u_0, v_0) = p$. Then*

$$T_pS = \operatorname{Im} D_{(u_0,v_0)}\psi.$$

In particular, T_pS is a vector space of dimension 2.

Proof. The proof consists of the following steps.

Step 1. *We have* $\operatorname{Im} D_{(u_0,v_0)}\psi \subset T_pS$.

Assume $v \in \operatorname{Im} D_{(u_0,v_0)}\psi$. Then there exists a vector $w \in \mathbb{R}^2$ such that $D_{(u_0,v_0)}\psi(w) = v$. Consider the smooth curve $\beta\colon (-\varepsilon, \varepsilon) \to \mathrm{V}$

$$\beta(t) = (u_0, v_0) + t \cdot w.$$

Then $\gamma(t) := \psi \circ \beta(t)$ is a smooth curve in S such that

$$\gamma(0) = \psi(\beta(0)) = \psi(u_0, v_0) = p \qquad \text{and} \qquad \dot\gamma(0) = D_{(u_0,v_0)}\psi(w) = v.$$

Hence, $v \in T_pS$.

Step 2. $T_pS \subset \operatorname{Im} D(u_0, v_0)\psi$

If $v \in T_pS$, then there exists $\gamma\colon (-\varepsilon, \varepsilon) \to S$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. Can assume $\operatorname{Im}\gamma \subset \mathrm{U}$ by choosing ε smaller if necessary. If $\varphi = \psi^{-1}$, then $\beta(t) := \varphi \circ \gamma(t)$ is a smooth curve in $\mathrm{V} \subset \mathbb{R}^2$ such that $\beta(0) = (u_0, v_0)$. Denote $w := \dot\beta(0) \in \mathbb{R}^2$. Then we have

$$\begin{aligned} v = \dot\gamma(0) = \frac{d}{dt}\Big|_{t=0} \big(\psi_0 \circ \beta\big)(t) &= \big(D_{(u_0,v_0)}\psi\big)\,\big(\dot\beta(0)\big) \\ = D_{(u_0,v_0)}\psi(w) &\in \operatorname{Im} D_{(u_0,v_0)}\psi. \end{aligned}$$

Step 3. $\dim T_pS = 2$.

This follows immediately from the injectivity of $D_{(u_0,v_0)}\psi$. □

Proposition 1.53. *Pick $p \in S$ and recall that there exists a neighbourhood $W \subset \mathbb{R}^3$ of p and a smooth function $\varphi\colon W \to \mathbb{R}$ such that*

$$S \cap W = \{q \in W \mid \varphi(q) = 0\} \quad \text{and} \quad \nabla\varphi(q) \neq 0 \quad \forall\, q \in W.$$

Then $T_pS = \nabla\varphi(p)^\perp$.

Proof. If γ is any curve in S through p, then

$$\varphi \circ \gamma(t) = 0 \quad \forall t \qquad \Longrightarrow \qquad \left.\frac{d}{dt}\right|_{t=0} \varphi(\gamma(t)) = 0.$$

Therefore, we obtain

$$0 = \left.\frac{d}{dt}\right|_{t=0} \varphi(\gamma(t)) = \langle \nabla\varphi(p), \dot\gamma(0)\rangle \qquad \Longrightarrow \qquad T_pS \subset \nabla\varphi(p)^{\perp}.$$

Since both T_pS and $\nabla\varphi(p)^{\perp}$ are two-dimensional, these spaces must be equal in fact. □

Example 1.54. Set $\varphi(x,y,z) = \big(x^2+y^2+z^2-1\big)/2$. Then $\varphi^{-1}(0) = S^2$ and

$$\nabla\varphi(p) = p \neq 0 \ \text{ if } \ p \in S^2 \qquad \Longrightarrow \qquad T_pS^2 = p^{\perp}.$$

This is consistent with Example 1.51.

Example 1.55. Set $\varphi(x,y,z) = \big(x^2+y^2-z^2-1\big)/2$. If $p=(x,y,z) \in H =: \varphi^{-1}(0)$, then $\nabla\varphi(p) = (x,y,-z) \neq 0$ and therefore

$$T_pH = (x,y,-z)^{\perp} = \big\{ v = (v_1,v_2,v_3) \in \mathbb{R}^3 \mid xv_1 + yv_2 - zv_3 = 0 \big\}.$$

Example 1.56. Set $\varphi(x,y,z) := \big(x^2+y^2-1\big)/2, \quad C = \varphi^{-1}(0) \ni p = (x,y,z)$. Then

$$T_pC = \big\{ v = (v_1,v_2,v_3) \mid xv_1 + yv_2 = 0, v_3 \text{ is arbitrary} \big\}.$$

1.5 The differential of a smooth map

Just as in calculus of several variables, we wish to study smooth functions, or, more generally, smooth maps, by approximating those by linear ones. This leads to the concept of the differential, which we define first for the case of functions. The more general case of smooth maps is considered below.

Definition 1.57 (Differential of a smooth function)**.** Let S be a surface and $f \in C^\infty(S)$. Define a map $d_pf\colon T_pS \to \mathbb{R}$ as follows: for $v \in T_pS$ choose a smooth curve γ throught p with $\dot\gamma(0) = v$ and set

$$d_pf(v) = \left.\frac{d}{dt}\right|_{t=0} f \circ \gamma(t). \tag{1.58}$$

Proposition 1.59. *d_pf is a well-defined linear map.*

Proof. Pick a parametrization $\psi\colon \mathrm{V} \to \mathrm{U} \ni p$. Without loss of generality we can assume that $\psi^{-1}(p) = 0 \in \mathrm{V}$.

If γ_1 and γ_2 are two curves through p such that $\dot\gamma_1(0) = v = \dot\gamma_2(0)$, then for $\beta_j := \psi^{-1} \circ \gamma_j$ we have

$$\gamma_j(t) = \psi \circ \beta_j(t) \qquad \Longrightarrow \qquad v = D_0\psi\big(\dot\beta_1(0)\big) = D_0\psi\big(\dot\beta_2(0)\big).$$

Since $D_0\psi$ is injective, we obtain $\dot\beta_1(0) = \dot\beta_2(0) =: w$. Furthermore,

$$\left.\frac{d}{dt}\right|_{t=0} f \circ \gamma_1(t) = \left.\frac{d}{dt}\right|_{t=0} \big(f \circ \psi \circ \psi^{-1} \circ \gamma_1(t)\big) = \left.\frac{d}{dt}\right|_{t=0} \big(F \circ \beta_1(t)\big) = D_0F(w).$$

Likewise, we obtain

$$\left.\frac{d}{dt}\right|_{t=0} f\circ\gamma_2\left(t\right) = D_0F\left(w\right) \qquad\Longrightarrow\qquad \left.\frac{d}{dt}\right|_{t=0}\left(f\circ\gamma_1\left(t\right)\right) = \left.\frac{d}{dt}\right|_{t=0}\left(f\circ\gamma_2\left(t\right)\right).$$

Hence, d_pf is well-defined and, moreover, we have the equality

$$d_pf\circ D_0\psi = D_0F,$$

where $F := f\circ\psi$ is the coordinate representation of f. Since both $D_0\psi$ and D_0F are linear, so is d_pf. □

Exercise 1.60. Think of $\mathbb{R}^2$ as a surface in $\mathbb{R}^3$ (for example, as $\mathbb{R}^2\times\{0\}\subset\mathbb{R}^3$). Let $f\colon\mathbb{R}^2\to\mathbb{R}^2$ be any smooth map. Show that the differential of f in the sense of Definition 1.57 coincides with the one known from the analysis course.

Exercise 1.61. If $h\in C^\infty\left(\mathbb{R}^3\right)$ and $f = h\big|_S$, then for all $p\in S$ we have

$$d_pf = D_ph\big|_{T_pS}.$$

Definition 1.62. A point $p\in S$ is called critical for $f\in C^\infty\left(S\right)$, if $d_pf = 0$, that is $d_pf\left(v\right) = 0$ for all $v\in T_pS$.

Proposition 1.63. *If p is a point of local maximum (minimum) for f, then p is critical for f.*

Proof. If p is a point of local maximum for f, then for any curve γ through p, 0 is a point of local maximum for $f\circ\gamma$. Hence, $\frac{d}{dt}\big|_{t=0}f\circ\gamma(t) = 0$. □

Proposition 1.64. *Let $h,\varphi\in C^\infty\left(\mathbb{R}^3\right)$. Assume $\nabla\varphi\left(p\right)\neq 0$ for any $p\in S = \varphi^{-1}\left(0\right)$. If $p\in S$ is a point of local maximum for $f = h\big|_S$, then*

$$\nabla h\left(p\right) = \lambda\nabla\varphi\left(p\right) \tag{1.65}$$

for some $\lambda\in\mathbb{R}$.

Proof. Our hypothesis implies that S is a surface and $T_pS = \big(\nabla\varphi(p)\big)^\perp$, see Example 1.9 and Proposition 1.53. Hence,

$$d_pf = 0 \qquad\Longleftrightarrow\qquad D_ph\big|_{T_pS} = 0 \qquad\Longleftrightarrow\qquad \langle v,\nabla h\left(p\right)\rangle = 0\quad\forall\, v\in T_pS.$$

In other words, $\nabla h(p)$ is orthogonal to T_pS. However, $T_pS^\perp$ is one-dimensional and contains $\nabla\varphi(p)\neq 0$. This implies (1.65). □

Remark 1.66. This proof is in a sense more conceptual than the proof of Theorem 1.6.

More generally, for any $f\in C^\infty\left(S;\mathbb{R}^n\right)$ the differential $d_pf\colon T_pS\to\mathbb{R}^n$ is defined by (1.58) too. This yields immediately the following: If f is written in components as $f = (f_1,\dots,f_n)$, then d_pf can be written in components as

$$d_pf = \big(d_pf_1,\dots,d_pf_n\big).$$

Also, the differential is well-defined for maps $f\colon\mathbb{R}^n\to S$ and is a linear map of the form $d_pf\colon\mathbb{R}^n\to T_{f(p)}S$. For maps $f\colon S_1\longrightarrow S_2$ between surfaces we define

$$d_pf\colon T_pS_1\longrightarrow T_{f(p)}S_2$$

essentially by the same rule: If $\dot\gamma\left(0\right) = v\in T_pS_1$, then $d_pf\left(v\right) := \frac{d}{dt}\big|_{t=0}\left(f\circ\gamma\left(t\right)\right)$. This yields again a well-defined linear map as the reader can easily check.

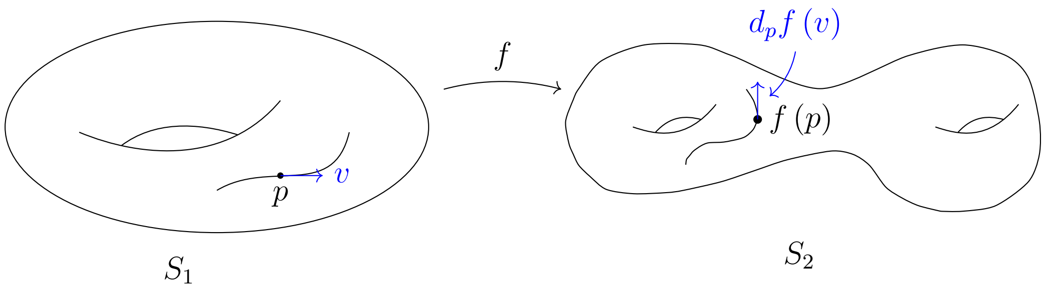


Figure 1.12: The differential of a smooth map

Proposition 1.67. *Let S_1, S_2, S_3 be smooth surfaces. For any smooth maps $f\colon S_1 \to S_2$ and $g\colon S_2 \to S_3$ and any point $p \in S_1$ we have*

$$D_p\,(g \circ f) = D_{f(p)}g \circ D_pf.$$

This also holds if any of S_i is replaces by an open subset of $\mathbb{R}^n$.

Proof. Let γ_1 be any smooth curve in S_1 through p. Denote $\gamma_2 = f \circ \gamma$, which is a smooth curve in S_2 through $f\,(p)$. If $\dot\gamma_1\,(0) = v_1$, then $v_2 := \dot\gamma_2\,(0) = D_pf\,(v_1)$ by the definition of D_pf. Hence,

$$D_p\,(g \circ f)\,(v_1) = \frac{d}{dt}\Big|_{t=0} \Big(g \circ \underbrace{f \circ \gamma_1}_{\gamma_2}(t)\Big) = \frac{d}{dt}\Big|_{t=0} \left(g \circ \gamma_2\,(t)\right) = D_{f(p)}g\,(v_2)$$
$$= D_{f(p)}g\,\big(D_pf\,(v_1)\big).$$

$\square$

Corollary 1.68. *If $f\colon S_1 \to S_2$ is a diffeomorphism, then $d_pf\colon T_pS_1 \to T_{f(p)}S_2$ is an isomorphism for any $p \in S_1$.* $\square$

Definition 1.69. A map $f\colon S_1 \to S_2$ is called *a local diffeomorphism* if for any $p \in S_1$ there exists a neighbourhood $\mathrm{U}_1 \subset S_1$ and a neighbourhood $\mathrm{U}_2 \subset S_2$ of $f\,(p)$ such that $f\colon \mathrm{U}_1 \to \mathrm{U}_2$ is a diffeomorphism.

Theorem 1.70. *Let $f\colon S_1 \to S_2$ be a smooth map such that $d_pf\colon T_pS_1 \to T_{f(p)}S_2$ is an isomorphism for all $p \in S_1$. Then f is a local diffeomorphism.*

Proof. Pick any $p \in S_1$ and parametrizations $\psi_1\colon \mathrm{V}_1 \to W_1 \subset S_1$ and $\psi_2\colon \mathrm{V}_2 \to W_2 \subset S_2$. Without loss of generality we can assume that $\psi_1\,(0) = p$ and $\psi_2\,(0) = f\,(p)$.

Recall that the coordinate representation of f is $F = \psi_2^{-1} \circ f \circ \psi_1$, see Fig. 1.13. Hence, by Proposition 1.67 we obtain $d_0F = d_{f(p)}\psi_2^{-1} \circ d_pf \circ d_0\psi_1$. Furthermore, since all of the following linear maps

$$d_0\psi_1\colon \mathbb{R}^2 \longrightarrow T_pS_1, \qquad d_{f(p)}\psi_2^{-1}\colon T_{f(p)}S_2 \longrightarrow \mathbb{R}^2, \qquad \text{and} \qquad d_pf\colon T_pS_1 \to T_pS_2$$

are isomorphisms, we conclude that d_0F is an isomorphism too.

From the analysis course it is known that there exists a neighbourhood $\widetilde{\mathrm{V}}_1 \subset \mathrm{V}_1$ of the origin and a neighbourhood $\widetilde{\mathrm{V}}_2 \subset \mathrm{V}_2$ of the origin such that $F\colon \widetilde{\mathrm{V}}_1 \to \widetilde{\mathrm{V}}_2$ is a diffeomorphism. Denoting $\mathrm{U}_1 = \psi_1\big(\widetilde{\mathrm{V}}_1\big)$ and $\mathrm{U}_2 = \psi_2\big(\widetilde{\mathrm{V}}_2\big)$, we have

$$f\big|_{\mathrm{U}_1} = \psi_2 \circ F \circ \psi_1^{-1}\big|_{\mathrm{U}_1}\colon \mathrm{U}_1 \to \mathrm{U}_2$$

is a diffeomorphism, since it is a composition of diffeomorphisms. $\square$

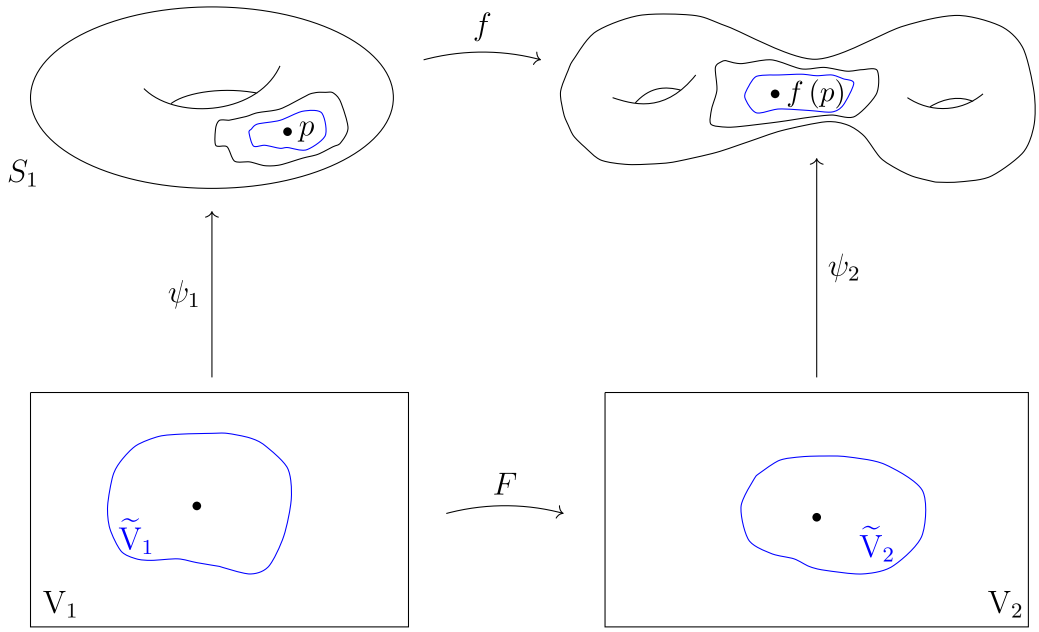


Figure 1.13: Illustration for the proof of Theorem 1.70

Remark 1.71. It follows from the proof of Theorem 1.70, that

$$d_p f = d_0\psi_2 \circ d_0 F \circ d_p\psi_1^{-1},$$

where both $d_0\psi_2$ and $d_p\psi_1^{-1}$ are linear isomorphisms.

In particular, this implies that the following holds:

- $d_p f$ is injective $\iff$ $D_{\psi_1(p)}F$ is injective;
- $d_p f$ is surjective $\iff$ $D_{\psi_1(p)}F$ is surjective;
- $d_p f$ is an isomorphism $\iff$ $D_{\psi_1(p)}F$ is an isomorphism.

Definition 1.72. For $f \in C^\infty(S_1; S_2)$ a point $p \in S_1$ is called *a critical point* of f if $d_p f$ is not surjective.

Since $\dim T_pS_1 = \dim T_{f(p)}S_2$, a simple argument from linear algebra yields:

$$d_p f \text{ is non-surjective} \iff d_p f \text{ is non-injective} \iff d_p f \text{ is not an isomorphism.} \quad (1.73)$$

Notice, however, that Definition 1.72 makes sense in more general situations where, for example, the target S_2 (and/or the source S_1) is replaced by $\mathbb{R}^n$. However, (1.73) is false in general for those more general cases.

To see that Definition 1.72 coincides with the previous one in the case of function, suppose p is a critical point of a smooth function $f\colon S_1 \to \mathbb{R}$ in the sense of Definition 1.72. If there exists $v \in T_pS_1$ such that $d_pf(v) \neq 0$, then the linearity of d_pf yields immediately that d_pf is surjective. Hence, d_pf is non-surjective if and only if it vanishes, cf. Definition 1.62.

Definition 1.74. A point $q \in S_2$ is called a *regular value* of f, if any $p \in f^{-1}(q)$ is a regular (that is non-critical) point of f, i.e., if for all $p \in f^{-1}(q)$ the differential d_pf is surjective.

The argument demonstrating (1.73) yields also the following:

$$d_p f \text{ is surjective} \iff d_p f \text{ is injective} \iff d_p f \text{ is an isomorphism.}$$

Example 1.75. Identify $\mathbb{C}$ with $\mathbb{R}^2$ and consider the map $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z^n$, where $n \in \mathbb{Z}, n \geq 2$. It is known from analysis that $d_z f\colon \mathbb{C} \to \mathbb{C}$ can be identified with the map $h \mapsto f'(z) \cdot h$. Hence, z is critical if and only if $f'(z) = 0 \Leftrightarrow nz^{n-1} = 0 \Leftrightarrow z = 0$. Hence, f has a single critical point $z = 0$ and a single critical value, the zero. All other points are regular and any non-zero value is also regular.

Viewing f as a map $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$, we obtain an example of a local diffeomorphism, which is not a diffeomorphism (assuming $n \geq 2$).

Theorem 1.76 (The fundamental theorem of algebra)**.** *Let $q(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$ with complex coefficients. Then p has at least one complex root.*

Proof. First recall that the map $f\colon S^2 \to S^2$,

$$f(p) = \begin{cases} N & p = N, \\ \psi_N \circ q \circ \psi_N^{-1}, & p \neq N, \end{cases}$$

is smooth. Indeed, the details of this claim are spelled on Page 14. The rest of the proof consists of the following steps.

Step 1. *f has at most n critical points (values).*

Indeed, a point $p \in S^2 \setminus \{N\}$ is critical for f if and only if $z := \psi_n(p)$ is critical for q. Hence, in this case $q'(z) = 0$, that is z is a root of the polynomial $nz^{n-1} + (n-1)\, a_{n-1} z^{n-2} + \ldots + a_1$, which can have at most $(n-1)$ roots.

Step 2. *Denote by $R(f)$ the set of regular values of f. Then for any $r \in R(f)$ the set $f^{-1}(r)$ is finite and the map $R(f) \to \mathbb{Z}_{\geq 0}, r \mapsto \# f^{-1}(r)$ is constant.*

Pick any $r \in R(f)$ and any $p \in f^{-1}(r)$. Then $f(p) = r$ and $d_p f$ is an isomorphism. Hence, by Theorem 1.70 there exists a neighbourhood U_p of p and a neighbourhood W_r such that $f\colon \mathrm{U}_p \to W_r$ is a diffeomorphism. In particular, $f^{-1}(r) \cap \mathrm{U}_p = \{p\}$, that is $f^{-1}(r)$ is discrete. Since $f^{-1}(r)$ is a closed subset of S^2, $f^{-1}(r)$ is compact. But a compact discrete set must be finite.

Denote $f^{-1}(r) = \{p_1, \ldots p_m\}$ and the corresponding neighbourhoods $\mathrm{U}_1, \ldots \mathrm{U}_m$ and $\mathrm{W}_1, \ldots \mathrm{W}_m$. Set $\mathrm{W} := \mathrm{W}_1 \cap \ldots \cap \mathrm{W}_m$ and $\widetilde{\mathrm{U}}_j := f^{-1}(\mathrm{W}) \cap \mathrm{U}_j$. Then for each $j \leq m$ the map $f\colon \widetilde{U}_j \to W$ is a diffeomorphism. In particular, for all $r' \in W$ there exists a unique $p_j' \in \widetilde{\mathrm{U}}_j$ such that $f\left(p_j'\right) = r'$. Hence, $\# f^{-1}(r') \geq \# f^{-1}(r)$ for all $r' \in W$.

Furthermore, I claim that in fact $\# f^{-1}(r') = \# f^{-1}(r)$ for all r' contained in some neighbourhood of r. Indeed, arguing by contradiction, assume that there is a sequence r_i' converging to r with the following property: for each i there is some $p_i' \in f^{-1}(r_i')$ such that p_i' is not contained in any of $\widetilde{\mathrm{U}}_1, \ldots, \widetilde{\mathrm{U}}_m$. By the compactness of S^2, a subsequence of p_i' converges to some p. Then, by the continuity of f we must have $f(p) = r$ so that $p = p_j$ for some $j \in \{1, \ldots, m\}$. But then a subsequence of p_i' must be contained in $\widetilde{\mathrm{U}}_j$. This is a contradiction.

Thus, the function

$$R(f) \longrightarrow \mathbb{Z}, \quad r \longmapsto \# f^{-1}(r) \tag{1.77}$$

is locally constant. However $R(f)$ is the complement of a finite number of points in S^2, hence connected. Therefore (1.77) is (globally) constant.

Step 3. *We prove this theorem.*

Pick any pairwise distinct points $p_1, \dots, p_{n+1} \in S^2 \backslash \{N\}$ such that $f(p_1), \dots, f(p_{n+1})$ are also pairwise distinct. Since f has at most n critical values, at least one of those points is a regular value of f and (1.77) does not vanish at this point. Hence, (1.77) vanishes nowhere on $R(f)$.

If the south pole S is a critical value of f, then $f^{-1}(S) \neq \varnothing$, since $f^{-1}(S)$ contains a critical point. However,

$$f^{-1}(S) \neq \varnothing \quad \Longleftrightarrow \quad q^{-1}(0) \neq \varnothing.$$

If S is a regular value, then Step 2 yields $\# f^{-1}(S) \geq 1$. This yields in turn $q^{-1}(0) \neq \varnothing$, which finishes this proof. $\square$

1.6 Orientability

Let $S \subset \mathbb{R}^3$ be a (smooth) surface.

Definition 1.78. A (smooth) map $v \colon S \to \mathbb{R}^3$ is called a (smooth) tangent vector field on S, if $v(p) \in T_pS$ for all $p \in S$.

Definition 1.79. A (smooth) map $n \colon S \to \mathbb{R}^3$ is called a (smooth) *normal field* on S, if $n(p) \perp T_pS$ for all $p \in S$.

Example 1.80. Set $S = S^2$, $n(x) = x$. Then n is a normal vector field on S^2.

Lemma 1.81. *Let $\psi \colon \mathrm{V} \to \mathrm{U} \subset S$ be a parametrization. Then U admits a unit normal field n on U, that is $n(p) \perp T_pS$ and $|n(p)| = 1$ holds for all $p \in \mathrm{U}$.*

Proof. Since ψ is a parametrization, for any $p \in \mathrm{U}$ there exists $q \in \mathrm{V}$ such that $\psi(q) = p$ and $D_q\psi \colon \mathbb{R}^2 \to T_pS = \operatorname{Im}(D_q\psi)$ is an isomorphism. Hence, $D_q\psi$ maps a basis of $\mathbb{R}^2$ onto a basis of T_pS. In particular, the image of the standard basis $(\partial_u\psi, \partial_v\psi)\big|_q$ is a basis of T_pS.

Define

$$n(p) = \frac{\partial_u\psi \times \partial_v\psi}{|\partial_u\psi \times \partial_v\psi|},$$

where "$\times$" means the cross-product in $\mathbb{R}^3$. This is well-defined, since $\partial_u\psi \times \partial_v\psi \neq 0$. $\square$

Exercise 1.82. Check that n is *a smooth* normal field on U.

Lemma 1.83. *If S is connected, then there are at most 2 non-equal unit normal fields on S.*

Proof. Let n_1 and n_2 be unit normal fields. Since for any $p \in S$ both $n_1(p)$ and $n_2(p)$ are orthogonal to T_pS and $|n_1|(p) = |n_2(p)|$, we must have $n_2(p) = \pm n_1(p)$.

Denote $S_\pm := \{p \in S \mid n_2(p) = \pm n_1(p)\}$. Then both S_+ and S_- are closed and $S = S_+ \cup S_-$. Hence, either

$$\begin{aligned} S_+ &= \varnothing \quad &\Longleftrightarrow& \quad n_2(p) = -n_1(p) \quad &\text{for any } p \in S \quad \text{or} \\ S_- &= \varnothing \quad &\Longleftrightarrow& \quad n_2(p) = +n_1(p) \quad &\text{for any } p \in S. \end{aligned}$$

$\square$

Definition 1.84. A surface S is said to be *orientable*, if S admits a unit normal field.

It should be intuitively clear that any unit normal field "selects a side" of the surface. A choice of the unit normal field ("a side of S") is called an orientation of S. Thus, any surface S admits at most 2 distinct orientations.

Proposition 1.85 (Preimages are orientable)**.** *If* 0 *is a regular value of* $\varphi\colon \mathbb{R}^3 \to \mathbb{R}$*, then* $S := \varphi^{-1}(0)$ *admits a unit normal field.*

Here, just like in the Definition 1.74, 0 is said to be the regular value of φ if for any $p \in S = \varphi^{-1}(0)$ we have

$$D_p\varphi\colon \mathbb{R}^3 \longrightarrow \mathbb{R} \text{ is surjective} \qquad \Longleftrightarrow \qquad \nabla\varphi(p) \neq 0,$$

since $D_p\varphi(v) = \langle \nabla\varphi(p), v\rangle$, where $v \in \mathbb{R}^3$.

Proof. Since $T_pS = \nabla\varphi(p)^\perp$, we see that $\nabla\varphi$ is a normal field. Since 0 is a regular value of φ, $\nabla\varphi$ vanishes nowhere on S. Hence, $n(p) := \frac{\nabla\varphi(p)}{|\nabla\varphi(p)|}$ is a unit normal field. $\square$

Remark 1.86. In the definition of orientability, it is only important, that the normal field exists, is non-vanishing and continuous. Smoothness can be deduced from this.

Example 1.87 (A non-example: the Möbius band)**.** One can obtain the Möbius band from the strip by gluing the opposite sides as shown on the figure.

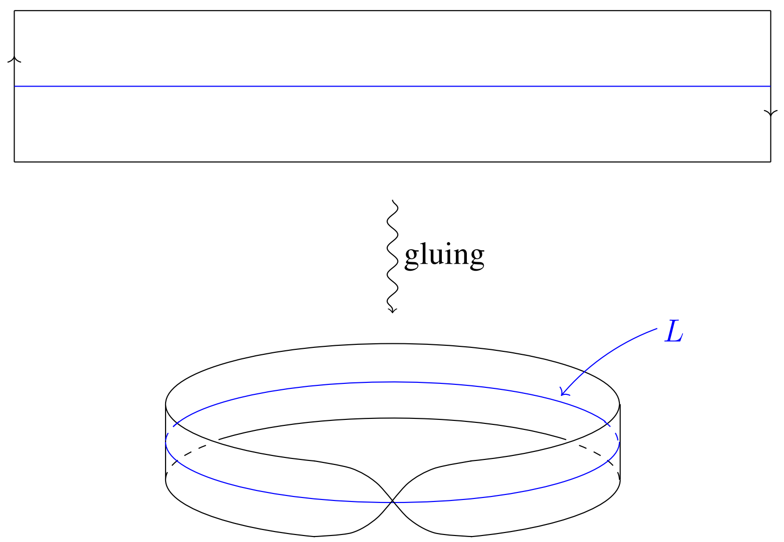


Figure 1.14: The Möbius band from the strip

More formally, the Möbius band is the image of the map

$$\begin{aligned}
&\Psi\colon\ [0,2\pi] \times (-1,1) \longrightarrow \mathbb{R}^3,\\
&\Psi(u,v) = \left(\left(2 - v\sin\frac{u}{2}\right)\sin u, \left(2 - v\sin\frac{u}{2}\right)\cos u, v\cos\frac{u}{2}\right).
\end{aligned}$$

Exercise 1.88. Show that the image of Ψ is a surface indeed.

To see that the Möbius band is non-orientable, recall that we showed in Lemma 1.81 that any point on a surface admits an orientable neighbourhood U. Moreover, it follows from the proof that given $0 \neq n_0 \perp T_{p_0}S$ at some $p_0 \in \mathrm{U}$, there is a unique orientation n of U such that $n(p_0) = \frac{n_0}{|n_0|}$. With this understood, for all $p \in L$ pick an orientable neighbourhood U_p. Since L is compact, there is a finite collection $\mathrm{U}_1, \ldots, \mathrm{U}_n$ covering L. Choose a point $p_1 \in L \cap \mathrm{U}_1$ and a vector $n_1 \in T_{p_1}S^\perp$, $|n_1| = 1$. This determines uniquely a normal field n on U_1 such that $n(p_1) = n_1$. If $\mathrm{U}_2 \cap \mathrm{U}_1 \neq \varnothing$, then there exists a unique smooth extension of n to $\mathrm{U}_1 \cup \mathrm{U}_2$. After finitely many steps we obtain a normal field n on $\mathrm{U}_1 \cup \ldots \cup \mathrm{U}_n \supset L$. However, as one travels once along L, this normal field must change its direction, that is $n(p_1) = -n(p_1)$, which is impossible. Hence, the Möbius band does not admit a unit normal field, that is the Möbius band is non-orientable.

Let S be a surface.

Definition 1.89. A collection $\mathcal{A} = \Big\{ (\psi_a, \mathrm{V}_a, \mathrm{U}_a) \mid a \in A \Big\}$ of parametrizations of S is said to be an atlas, if $\bigcup_{a\in A} \mathrm{U}_a = S$.

Recall that for $a, b \in A$ the map

$$\theta_{ab} \coloneqq \psi_a^{-1} \circ \psi_b \colon \mathrm{V}_{ab} = \psi_b^{-1}(\mathrm{U}_a \cap \mathrm{U}_b) \longrightarrow \mathbb{R}^2$$

is called the change of coordinates map.

Definition 1.90. An atlas $\mathcal{A}$ on S is said to be oriented, if $\det\big(D_{(u,v)}\theta_{ab}\big) > 0$ for any $(u,v) \in \mathrm{V}_{ab}$.

Example 1.91. For $S = S^2, \mathcal{A} = \Big\{ \big(\psi_N, \mathbb{R}^2, S^2\backslash\{N\}\big), \big(\psi_S, \mathbb{R}^2; S^2\backslash\{S\}\big) \Big\}$ is an atlas. We have

$$\theta_{SN}(u,v) = \frac{1}{u^2+v^2}(u,v)$$

A computation yields $\det(D\theta_{SN}) < 0$, so that $\mathcal{A}$ is *not* an oriented atlas.

Consider, however

$$\mathcal{B} = \Big\{ \big(\psi_N, \mathbb{R}^2, S^2\backslash\{N\}\big), \big(\widehat{\psi}_S, \mathbb{R}^2, S^2\backslash\{S\}\big) \Big\},$$

where $\widehat{\psi}_S(u,v) = \psi_S(-u,v) = \psi_S \circ \sigma(u,v)$, where $\sigma(u,v) = (-u,v)$. Then

$$\widehat{\theta}_{SN} = \widehat{\psi}_S^{-1} \circ \psi_N = (\psi_S \circ \sigma)^{-1} \circ \psi_N = \sigma^{-1} \circ \theta_{SN} = \sigma \circ \theta_{SN},$$

since $\sigma^{-1} = \sigma$. By the linearity of σ, we have $D\widehat{\theta}_{SN} = \sigma \circ D\theta_{SN}$, which yields

$$\det D\widehat{\theta}_{SN} = \det\sigma \cdot \det D\theta_{SN} > 0,$$

since $\det\sigma = -1$ and $\det D\theta_{SN} < 0$. Thus, $\mathcal{B}$ is an oriented atlas on S^2.

Proposition 1.92. *A surface S is orientable if and only if S admits an oriented atlas.*

Proof. The proof consists of the following steps.

Step 1. *If S is orientable, then S admits an oriented atlas.*

Choose a unit normal field n on S and an atlas $\mathcal{A}$ on S. Define a new atlas $\mathcal{B}$ as follows: If $\psi_a \colon \mathrm{V}_a \to \mathrm{U}_a$ belongs to $\mathcal{A}$ and $\det\big(\partial_u\psi_a, \partial_v\psi_a, n(\psi_a(u,v))\big) > 0$, then $(\psi_a, \mathrm{V}_a, \mathrm{U}_a)$ belongs to $\mathcal{B}$. If $\det\big(\partial_u\psi_a, \partial_v\psi_a, n(\psi_a(u,v))\big) < 0$, then $\big(\psi_a \circ \sigma, \sigma(\mathrm{V}_a), \mathrm{U}_a\big) = \big(\widehat{\psi}_a, \widehat{\mathrm{V}}_a, \mathrm{U}_a\big)$ belongs to $\mathcal{B}$, where $\sigma \colon \mathbb{R}^2 \to \mathbb{R}^2, \sigma(u,v) = (-u,v)$. This yields

$$\det\Big(\partial_u\widehat{\psi}_a, \partial_v\widehat{\psi}_a, n\big(\widehat{\psi}_a(u,v)\big)\Big) = \det\Big(-\partial_u\psi_a, \partial_v\psi_a, n\big(\psi_a(u,v)\big)\Big) > 0.$$

Therefore, we obtain:

(a) Suppose both $\psi_a \colon \mathrm{V}_a \longrightarrow \mathrm{U}_a$ and $\psi_b \colon \mathrm{V}_b \longrightarrow \mathrm{U}_b$ belong to $\mathcal{B}$. Denote by (x_1, x_2) and (y_1, y_2) coordinates on V_a and V_b respectively. Write the transition map $\theta = \theta_{ab} \colon \mathrm{V}_b \longrightarrow \mathrm{V}_a$, which is defined on an open subset of V_b, in components as $\theta = (\theta_1, \theta_2)$. Then from $\psi_b = \psi_a \circ \theta$ we obtain

$$\begin{aligned}\partial_{y_1}\psi_b &= \partial_{x_1}\psi_a\left(\theta\left(y\right)\right)\partial_{y_1}\theta_1 + \partial_{x_2}\psi_a\left(\theta\left(y\right)\right)\partial_{y_1}\theta_2,\\ \partial_{y_2}\psi_b &= \partial_{x_2}\psi_a\left(\theta\left(y\right)\right)\partial_{y_2}\theta_1 + \partial_{x_2}\psi_a\left(\theta\left(y\right)\right)\partial_{y_2}\theta_2.\end{aligned}$$

In matrix notations this can be written more briefly as

$$\left(\partial_{y_1}\psi_b, \partial_{y_2}\psi_b\right) = \left(\partial_{x_1}\psi_a, \partial_{x_2}\psi_b\right) \cdot \partial_y\theta, \qquad \text{where} \quad \partial_y\theta = \begin{pmatrix}\partial_{y_1}\theta_1 & \partial_{y_2}\theta_1\\ \partial_{y_1}\theta_2 & \partial_{y_2}\theta_2\end{pmatrix}$$

is the Jacobi matrix of $\theta = \theta_{ab}$. Hence,

$$\left(\partial_{y_1}\psi_b, \partial_{y_2}\psi_b, n\right) = \left(\partial_{x_1}\psi_a, \partial_{x_2}\psi_a, n\right)\left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$\partial_y\theta$}} & 0\\ & & 0\\ \hline 0 & 0 & 1\end{array}\right),$$

which yields in turn

$$\det\left(\partial_{y_1}\psi_b, \partial_{y_2}\psi_b, n\right) = \det\left(\partial_{x_1}\psi_a, \partial_{x_2}\psi_a, n\right) \cdot \det\left(\begin{array}{c|c}\partial_y\theta & \\ \hline & 1\end{array}\right). \tag{1.93}$$

By the assumption, we have $\det\left(\partial_{y_1}\psi_b, \partial_{y_2}\psi_b, n\right) > 0$ and $\det\left(\partial_{x_1}\psi_a, \partial_{x_2}\psi_a, n\right) > 0$. Hence, using (1.93) and

$$\det\left(\begin{array}{c|c}\partial_y\theta & \\ \hline & 1\end{array}\right) = \det\left(\partial_y\theta\right)$$

we obtain $\det\left(\partial_y\theta\right) > 0$.

(b) If $\widehat{\psi}_a$ and $\widehat{\psi}_b$ belong to $\mathcal{B}$, essentially the same computation as above yields

$$\det\left(\partial_y\theta_{ab}\right) > 0.$$

Furthermore,

$$\psi_b = \psi_a \circ \theta_{ab} \qquad \Longrightarrow \qquad \psi_b \circ \sigma = \psi_a \circ \theta_{ab} \circ \sigma = \left(\psi_a \circ \sigma\right) \circ \sigma \circ \theta_{ab} \circ \sigma$$

where $\psi_b \circ \sigma = \widehat{\psi}_b$ and $\psi_a \circ \sigma = \widehat{\psi}_a$. Hence, the change of coordinates map between $\widehat{\psi}_a$ and $\widehat{\psi}_b$ is $\widehat{\theta}_{ab} := \sigma \circ \theta_{ab} \circ \sigma$. This yields

$$\det\left(\partial_y\widehat{\theta}_{ab}\right) = \det\sigma \cdot \det\partial_y\theta_{ab} \cdot \det\sigma = \left(\det\sigma\right)^2\det\partial_y\theta_{ab} > 0.$$

(c) Suppose finally that ψ_a and $\widehat{\psi}_b$ belong to $\mathcal{B}$. By the same argument as above, we obtain $\det\left(\partial_y\theta_{ab}\right) < 0$. If $\widehat{\theta}_{ab}$ denotes the change of coordinates between ψ_a and $\widehat{\psi}_b$, then

$$\widehat{\theta}_{ab} = \theta_{ab} \circ \sigma \qquad \Longrightarrow \qquad \det\partial_y\widehat{\theta}_{ab} = \det\left(\partial_y\theta_{ab}\right) \cdot \det\sigma > 0,$$

since both $\det\left(\partial_y\theta_{ab}\right)$ and $\det\sigma$ are negative.

Thus, $\mathcal{B}$ is an oriented atlas.

Step 2. *If S admits an oriented atlas, then S admits a unit normal field.*

Let $\mathcal{A}$ be an oriented atlas on S and $\psi_a \colon \mathrm{V}_a \to \mathrm{U}_a$ a parametrization from $\mathcal{A}$. If $\psi_a(q) = p \in \mathrm{U}_a$, define $n(p)$ by

$$n(p) = \left. \frac{\partial_u \psi_a \times \partial_v \psi_a}{|\partial_u \psi_a \times \partial_v \psi_a|} \right|_q .$$

Assume ψ_b is another parametrization from $\mathcal{A}$ such that $p \in \mathrm{U}_b$. Then $\psi_b = \psi_a \circ \theta$, where $\theta = \theta_{ab}$, so that

$$\left(\partial_{y_1}\psi_b, \partial_{y_2}\psi_b\right) = \left(\partial_{x_1}\psi_a, \partial_{x_2}\psi_b\right) \cdot \partial_y \theta \quad \Longrightarrow \quad \partial_{y_1}\psi_b \times \partial_{y_2}\psi_b = \det\left(\partial_y \theta\right) \cdot \partial_{x_1}\psi_a \times \partial_{x_2}\psi_b,$$

where $\det\left(\partial_y \theta\right) > 0$. Hence $n(p)$ does not depend on the choice of parametrization near p. Since n is smooth in a neighbourhood of p, n is smooth everywhere. □

1.7 Partitions of unity

Recall that the function $\lambda \colon \mathbb{R} \to \mathbb{R}$

$$\lambda(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ e^{-\frac{1}{t}} & \text{if } t > 0 \end{cases}$$

is smooth.

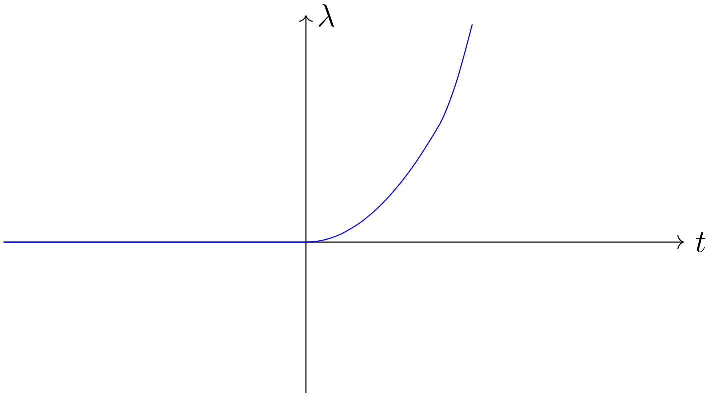


Figure 1.15: The graph of λ

For any fixed $r > 0$ and all $t \in \mathbb{R}$ we have

$$\lambda(t) + \lambda(r - t) > 0,$$

because $\lambda(t)$ is positive for $t > 0$ and $\lambda(r - t)$ is positive for $t < r$. Define

$$\widehat{\chi}_r(t) \coloneqq \frac{\lambda(r - t)}{\lambda(t) + \lambda(r - t)},$$

which is smooth everywhere on $\mathbb{R}$. Denote also

$$\chi_r(t) \coloneqq \chi_r(t - 1)$$

Lemma 1.94. *For any point $p \in \mathbb{R}^n$ and any neighbourhood $\mathrm{U} \ni p$ there exists a neighbourhood $\mathrm{V} \subset \mathrm{U}$ and $\rho \in C^\infty(\mathbb{R}^n)$ such that the following holds:*

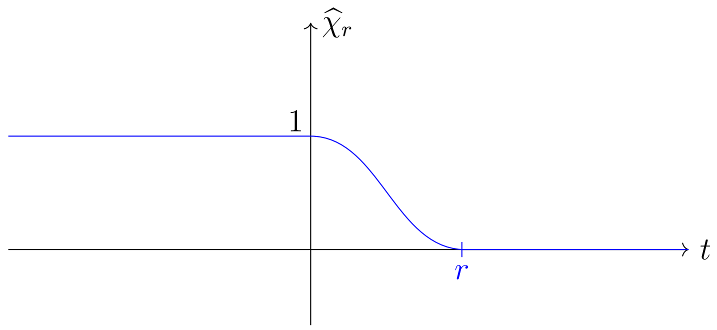


Figure 1.16: The graph of $\hat{\chi}_r$

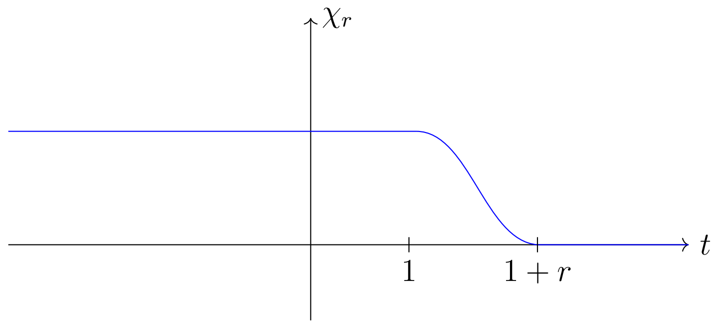


Figure 1.17: The graph of χ_r

- $0 \leq \rho(x) \leq 1$ *for all* $x \in \mathbb{R}^n$*;*
- $\rho\big|_{\mathrm{V}} \equiv 1$ *and* $\rho\big|_{\mathbb{R}^n \setminus \mathrm{U}} \equiv 0$*.*

Proof. For any $R > 0$, consider

$$\rho(x) := \chi_1\left(\frac{|x-p|}{R}\right).$$

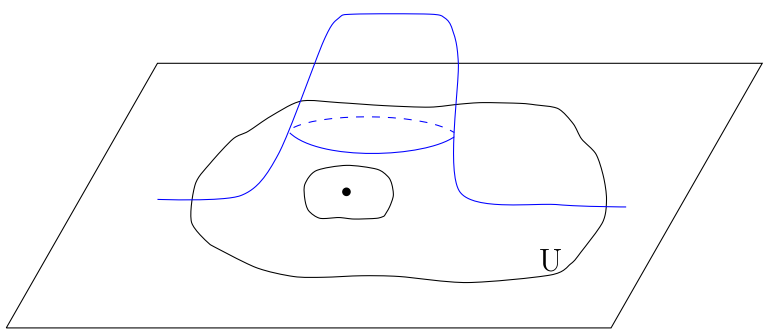


Figure 1.18: Schematic graph of ρ

If $B_{2R}(p) \subset \mathrm{U}$, then ρ vanishes outside of $B_{2R}(p)$, so vanishes outside of U. Also, $\rho(x) \equiv 1$ on $B_{2R}(p)$ and $\rho \in C^\infty$. Here $B_{2R}(p)$ is the ball of radius $2R$ centered at p. $\square$

Definition 1.95. For a continuous function f on a topological space X define *the support* of f by

$$\operatorname{supp} f := \overline{\{x \in X \mid f(x) \neq 0\}}.$$

Notice in particular, that for $x \notin \operatorname{supp} f$ we have $f(x) = 0$. However, a function may still have zeros in its support. For example, $\operatorname{supp}\lambda = [0, +\infty)$ so that $0 \in \operatorname{supp}\lambda$ and $\lambda(0) = 0$.

In fact, unwinding the definition in full details, we obtain that $x \in \operatorname{supp} f$ if and only if there exists a sequence $x_n \to x$ such that $f(x_n) \neq 0$. In other words,

$$x \notin \operatorname{supp} f \quad \iff \quad \exists \text{ a neighbourhood } \mathrm{U} \text{ of } x \text{ such that } f\big|_{\mathrm{U}} \equiv 0.$$

Example 1.96. If ρ is as in the above lemma, then $\operatorname{supp}\rho \subset \mathrm{U}$.

Example 1.97. For $f(x) = |x|^2 - 1$, $f\colon \mathbb{R}^n \to \mathbb{R}$, $\operatorname{supp} f = \mathbb{R}^n$.

Definition 1.98. *A (smooth) partition of unity* on $\mathbb{R}^n$ is a family of smooth functions $\{\rho_\alpha \mid \alpha \in A\}$ such that

(i) $0 \leq \rho_\alpha(x) \leq 1$ for all $x \in \mathbb{R}^n$ and all $\alpha \in A$;

(ii) For any $x \in \mathbb{R}^n$ the set $\big\{\alpha \in A \mid \rho_\alpha(x) \neq 0\big\}$ is finite;

(iii) $\sum_{\alpha\in A} \rho_\alpha(x) = 1$ for all $x \in \mathbb{R}^n$.

Remark 1.99. More precisely, (ii) in the above definition should be replaced by the following condition: $\forall\, x \in \mathbb{R}^n$ there exists a neighbourhood $\mathrm{V} \ni x$ such that the set $\{\alpha \in A \mid \operatorname{supp}\rho_\alpha \cap \mathrm{V} \neq \varnothing\}$ is finite. However, we consider mostly finite partitions of unity so that this condition (and therefore, also (ii)) will be satisfied automatically.

Example 1.100 (A partition of unity on $\mathbb{R}$)**.** Consider $\big\{\widehat{\rho}_j(x) \mid j \in \mathbb{Z}\big\}$, where $\widehat{\rho}_j(x) = \chi_1(|x - j|)$. Notice that $\operatorname{supp}\widehat{\rho}_j \subset [j-2, j+2]$ so that the function $\widehat{\rho}(x) := \sum_{j\in\mathbb{Z}} \widehat{\rho}_j(x)$ well-defined, smooth and positive everywhere on $\mathbb{R}$. Hence,

$$\big\{\rho_j = \widehat{\rho}_j \,/\, \widehat{\rho} \mid j \in \mathbb{Z}\big\}$$

is a partition of unity on $\mathbb{R}^1$.

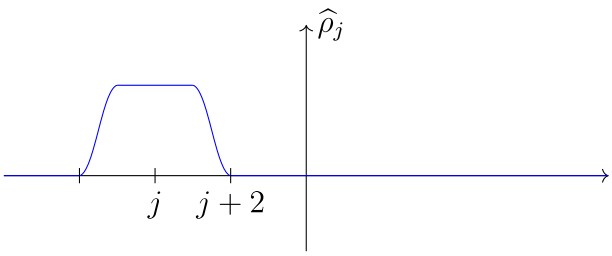


Figure 1.19: The schematic graph of ρ_j

Partitions of unity for surfaces are defined just like for $\mathbb{R}^n$.

Theorem 1.101 (Existence of a partition of unity)**.** *Let $\mathcal{U} = \{\mathrm{U}_\alpha \mid \alpha \in A\}$ be any open covering of a surface S. Then there exists a partition of unity $\{\rho_\beta \mid \beta \in B\}$ such that for each $\beta \in B$ there exists an $\alpha \in A$ so that*

$$\operatorname{supp}\rho_\beta \subset \mathrm{U}_\alpha.$$

Proof. The proof is given for compact surfaces only.

Step 1. *Let S be any surface. For any $p \in S$ and any open subset $W \subset S$ such that $p \in W$, there exist $\rho \in C^\infty(S)$ such that*

(i) $0 \le \rho(q) \le 1$ *for $q \in S$;*

(ii) $\operatorname{supp} \rho \subset W$*;*

(iii) There exists an open subset $X \subset W$ such that $p \in X$ and $\rho\big|_X \equiv 1$.

Let (U, φ) be a chart on S such that $\varphi(p) = 0 \in \mathrm{V} \subset \mathbb{R}^2$ and $\mathrm{U} \subset W$. Pick a function $\widehat{\rho} \in C^\infty(\mathbb{R}^2)$ such that $0 \le \widehat{\rho} \le 1$, $\widehat{\rho}\big|_{B_r(0)} \equiv 1$, and $\widehat{\rho}\big|_{\mathbb{R}^2 \setminus B_{2r}(0)} \equiv 0$ for some $r > 0$ such that $B_{2r}(0) \subset \mathrm{V}$. Define

$$\rho(p) := \begin{cases} \widehat{\rho} \circ \varphi(p) & \text{if } p \in \mathrm{U}, \\ 0 & \text{otherwise.} \end{cases}$$

Then ρ is smooth everywhere and with $X := \varphi^{-1}(B_r(0))$ satisfies *(i)*–*(iii)*.

Remark 1.102. Alternatively, one can first define a suitable function $\widetilde{\rho}$ on a neighbourhood of p in $\mathbb{R}^3$ and define ρ as the restriction of $\widetilde{\rho}$ to S.

Remark 1.103. Any function satisfying Properties *(i)*–*(iii) of Step 1* is called a bump function.

Step 2. *We prove this theorem assuming S is compact.*

Pick any U_α and any $p \in \mathrm{U}_\alpha$. By Step 1, there exists $X_{p,\alpha} \subset \mathrm{U}_\alpha$ and a function $\widehat{\rho}_{p,\alpha}$ satisfying *(i)*–*(iii)*.

Consider the family $\{X_{p,\alpha} \mid p \in S, \alpha \in A\}$, which is an open covering of S. By the compactness of S, there exists a finite subcovering $\{X_{p_1,\alpha_1}, \dots, X_{p_n,\alpha_n}\}$. To simplify notations, redenote $X_j := X_{p_j,\alpha_j}$ and $\widehat{\rho}_j := \widehat{\rho}_{p_j,\alpha_j}$ so that $\widehat{\rho}_j\big|_{X_j} \equiv 1$. Just as in Example 1.100, we have

$$\widehat{\rho}(p) := \sum_{j=1}^n \widehat{\rho}_j(p) > 0$$

for any $p \in S$. Then $\rho_j := \widehat{\rho}_j \,/\, \widehat{\rho}$ is a partition of unity on S. Moreover, $\operatorname{supp} \rho_j = \operatorname{supp} \widehat{\rho}_j \subset \mathrm{U}_{\alpha_j}$. □

Remark 1.104. A partition of unity as in the above theorem is called *subordinate* to $\mathcal{U}$.

Example 1.105. Consider the case $S = S^2$ with the covering $\mathcal{U} = \{S^2 \setminus \{N\}, S^2 \setminus \{S\}\}$. Albeit the above theorem yields a partition of unity subordinate to $\mathcal{U}$, we can construct this by hands as follows. Let ρ be a bump function on $\mathbb{R}^2$ such that $\rho\big|_{B_1(0)} \equiv 1$ and $\operatorname{supp} \rho \subset B_2(0)$. Define

$$\rho_N := \rho \circ \varphi_N \qquad \text{and} \qquad \rho_S := 1 - \rho_N.$$

Then $\{\rho_N, \rho_S\}$ is the partition of unity we are looking for.

1.8 Integration on surfaces

The aim of this section is to define a map $\int : C^\infty(S) \longrightarrow \mathbb{R}$ with "the usual" properties of the integral, e.g.

$$\int_S (\lambda f + \mu g) = \lambda \int_S f + \mu \int_S g \qquad \lambda, \mu \in \mathbb{R} \quad f, g \in C^\infty(S). \tag{1.106}$$

To this end, assume that S is compact and choose an atlas $\mathcal{A} = \{(\mathrm{U}_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ on S. Let $\{\rho_j \mid j = 1, \dots, J\}$ be a partition of unity on S such that $\operatorname{supp}\rho_j \subset \mathrm{U}_{\alpha_j} =: \mathrm{U}_j$. For any $f \in C^\infty(S)$ we have

$$f = f \cdot 1 = \sum_{j=1}^{J} f \cdot \rho_j = \sum_j f_j,$$

where $f_j := f \cdot \rho_j$ and $\operatorname{supp} f_j \subset \operatorname{supp}\rho_j \subset \mathrm{U}_j$. Hence, by (1.106) it suffices to define $\int_S f_j$, that is we want to define $\int_S f$ provided $\operatorname{supp} f \subset \mathrm{U}$, where (U, φ) is a chart.

Viewing φ as an identification between U and $\mathrm{V} \subset \mathbb{R}^2$, we can identify f with its coordinate representation

$$F := f \circ \varphi^{-1} = f \circ \psi \colon \mathrm{V} \longrightarrow \mathbb{R}.$$

Then F vanishes outside of $\varphi^{-1}(\operatorname{supp} f)$, which is compact.

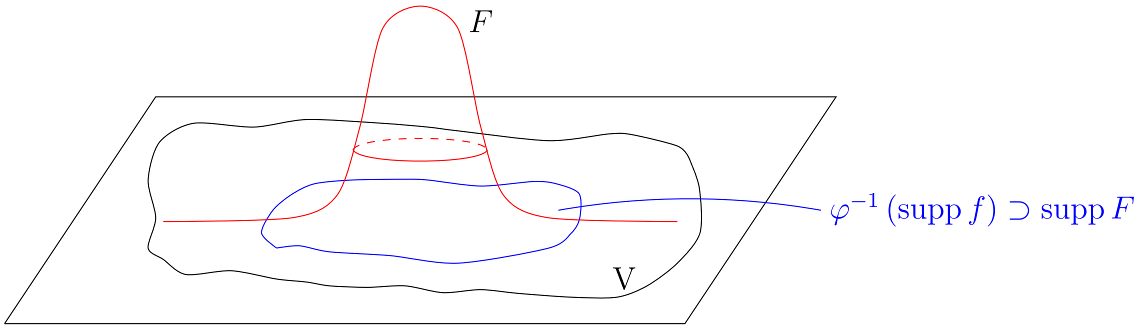


Figure 1.20: The coordinate representation of f

It is tempting to define

$$\int_S f := \int_{\mathbb{R}^2} F(u, v)\, du\, dv. \tag{1.107}$$

Notice that the integrand on the right hand side of the above equality vanishes outside of a compact set so that in fact we do not need to worry about the convergence of this integral. It may happen, however, that there is another chart $(\widehat{\mathrm{U}}, \widehat{\varphi})$ on S such that $\operatorname{supp} f \subset \widehat{\mathrm{U}}$. To show that $\int_S f$ is well-defined, we must show the equality

$$\int_{\mathbb{R}^2} F(u, v)\, du\, dv \overset{?}{=} \int_{\mathbb{R}^2} \widehat{F}(x, y)\, dx\, dy, \tag{1.108}$$

where $\widehat{F} = f \circ \widehat{\varphi}^{-1}$ is the coordinate representation of f with respect to $\widehat{\varphi}$.

Let $\theta = \varphi \circ \widehat{\varphi}^{-1} \Leftrightarrow (u, v) = \theta(x, y)$ denote the change of coordinates map. Then

$$\widehat{F} = f \circ \widehat{\varphi}^{-1} = f \circ \varphi^{-1} \circ \varphi \circ \widehat{\varphi}^{-1} = F \circ \theta,$$

so that (1.108) is equivalent to

$$\int_{\mathbb{R}^2} F(u, v)\, du\, dv \overset{?}{=} \int_{\mathbb{R}^2} F \circ \theta(x, y)\, dx\, dy$$

The last equality is false in general, since by a well-known theorem from analysis we have

$$\int_{\mathbb{R}^2} F(u, v)\, du\, dv = \int_{\mathbb{R}^2} F \circ \theta(x, y)\, |\det D\theta| dx\, dy.$$

Thus, our naïve approach to define $\int_S f$ by (1.107) does not work in general.

To solve this problem, recall the following fact. Suppose $\mathrm{V} \subset \mathbb{R}^3$ is a bounded open set such that $S := \partial \mathrm{V}$ is a smooth oriented surface. Then by the divergence theorem we have

$$\int_{\mathrm{V}} \operatorname{div} v = \int_S \langle v, n\rangle \, dS,$$

where n is the unit normal field pointing outwards. If $\psi = \psi(u, v)$ is a parametrization of S, the right hand side is defined by

$$\int \langle v, n\rangle |\partial_u \psi \times \partial_v \psi| \, du \, dv.$$

Following this hint, for $f \in C^\infty(S)$ with $\operatorname{supp} f \subset \mathrm{U}$, where U is a coordinate chart, we define

$$\int_S f := \int_{\mathbb{R}^2} F(u, v) \, |\partial_u \psi \times \partial_v \psi| du \, dv. \tag{1.109}$$

Then, if $\left(\widehat{\mathrm{U}}, \widehat{\varphi}\right)$ is another chart just like above and $\theta = \varphi \circ \widehat{\varphi}^{-1} = \psi^{-1} \circ \widehat{\psi}$, we have

$$\begin{aligned} \widehat{\psi} = \psi \circ \theta \qquad &\Longrightarrow \qquad \left(\partial_x \widehat{\psi}, \partial_y \widehat{\psi}\right) = (\partial_u \psi, \partial_v \psi) \cdot D\theta \\ &\Longrightarrow \qquad |\partial_x \widehat{\psi} \times \partial_y \widehat{\psi}| = |\partial_u \psi \times \partial_v \psi| \cdot |\det D\theta|. \end{aligned}$$

Hence, we have

$$\begin{aligned} \int_{\mathbb{R}^2} \widehat{F}(x, y) \, |\partial_x \widehat{\psi} \times \partial_y \widehat{\psi}| dx \, dy &= \int_{\mathbb{R}^2} F \circ \theta (x, y) \, |\partial_u \psi \times \partial_v \psi| \, |\det D\theta| \, dx \, dy \\ &= \int_{\mathbb{R}^2} F(u, v) \, |\partial_u \psi \times \partial_v \psi| du \, dv. \end{aligned}$$

That is (1.109) does not depend on the choice of the parametrization of S.

Definition 1.110. Let S be a compact surface and f a smooth function on S. Pick an atlas $\mathcal{U} = \{(\mathrm{U}_\alpha, \varphi_\alpha)\}$ and a finite partition of unity $\{\rho_j \mid 1 \leq j \leq J\}$ subordinate to $\mathcal{U}$. Denote by F_j the coordinate representation of $f_j := \rho_j \cdot f$. Then the integral of f over S is defined by

$$\int_S f := \sum_j \int_S f_j = \sum_j \int_{\mathbb{R}^2} F_j(u, v) \, |\partial_u \psi_j \times \partial_v \psi_j| \, du \, dv,$$

where $\operatorname{supp} \rho_j \subset \mathrm{U}_j = \mathrm{U}_{\alpha(j)}$ and $\psi_j = \varphi_{\alpha(j)}^{-1}$.

Proposition 1.111. *$\int_S f$ is well-defined, that is $\int_S f$ does not depend on the choice of an atlas.*

Proof. Let $\widehat{\mathcal{U}} = \left\{\left(\widehat{\mathrm{U}}_\beta, \widehat{\varphi}_\beta\right) \mid \beta \in B\right\}$ be another atlas on S. Choose a partition of unity $\{\mu_k \mid k = 1, \dots, K\}$ subordinate to $\widehat{\mathcal{U}}$. We need to show that

$$\sum_j \int_S (\rho_j f) \stackrel{?}{=} \sum_k \int_S (\mu_k f). \tag{1.112}$$

Notice that $\{\lambda_{jk} := \rho_j \lambda_k \mid j = 1, \dots, J, k = 1, \dots, K\}$ is also a partition of unity and $\operatorname{supp} \lambda_{jk} \subset \mathrm{U}_j \cap \widehat{\mathrm{U}}_k$.

With this understood, for a fixed j consider

$$\sum_{k=1}^{K}\int_S \lambda_{jk} f = \int_S\left(\rho_j\sum_{k=1}^{K}\mu_k f\right) = \int_S \rho_j f,$$

where the first equality follows by the linearity of the integral on the space of compactly supported functions on $\mathbb{R}^2$. Summing the above equality over j, we arrive at

$$\sum_{j=1}^{J}\sum_{k=1}^{K}\int_S \lambda_{jk} f = \sum_{j=1}^{J}\int_S\left(\rho_j\sum_{k=1}^{K}\mu_k f\right) = \sum_{j=1}^{J}\int_S \rho_j f.$$

Similarly, we have

$$\sum_{k=1}^{K}\sum_{j=1}^{J}\int_S \lambda_{jk} f = \sum_{k}\int_S\left(\mu_k\sum_{j=1}^{J}\rho_j f\right) = \sum_{k}\int_S \mu_k f.$$

Comparing the above two equalities we see that (1.112) holds indeed. $\square$

It follows immediately from the definition that $\int_S$ has the usual properties known from the analysis course, for example:

- $\int_S (\lambda f + \mu g) = \lambda\int_S f + \mu\int_S g$;
- $f \geq 0 \quad\Longrightarrow\quad \int_S f \geq 0$;
- $\int_S f = 0$ and $f \geq 0 \quad\Longrightarrow\quad f \equiv 0$

and so on, where in the last property I assume that f is at least continuous.

Example 1.113. Let $f\colon S^2 \longrightarrow \mathbb{R}$ be any (smooth) function. Let $\mathcal{U} = \big\{S^2\setminus\{N\}, S^2\setminus\{S\}\big\}$ be just as in Example 1.105. Choose $\varepsilon > 0$ and set

$$\rho_N^\varepsilon(p) := \rho\left(\varepsilon\varphi_N(p)\right), \qquad \text{and} \qquad \rho_S^\varepsilon := 1 - \rho_N^\varepsilon,$$

where ρ is just as in Example 1.105. Notice the following:

$$\begin{aligned}
\rho\Big|_{B_1(0)} \equiv 1 \quad &\Longrightarrow \quad \rho_N^\varepsilon\Big|_{\varphi_N^{-1}\left(B_{\varepsilon^{-1}}(0)\right)} \equiv 1,\\
\rho\Big|_{\mathbb{R}^2\setminus B_2(0)} \quad &\Longrightarrow \quad \rho_N^\varepsilon\Big|_{S^2\setminus\varphi_N^{-1}\left(B_{2\varepsilon^{-1}}(0)\right)} \equiv 0.
\end{aligned}$$

If $F_N = f\circ\psi_N$ and $F_S := f\circ\psi_S$ are coordinate representations of f, then by the definition of the integral we have

$$\begin{aligned}
\int_S f &= \int_{\mathbb{R}^2}\left(\rho_N^\varepsilon\circ\psi_N(u,v)\right)F_N(u,v)\,|\partial_u\psi_N\times\partial_v\psi_N|du\,dv\\
&\quad+\int_{\mathbb{R}^2}\left(\rho_S^\varepsilon\circ\psi_S(u,v)\right)\cdot F_S(u,v)\,|\partial_u\psi_S\times\partial_v\psi_S|du\,dv\\
&=\int_{\mathbb{R}^2}\rho\left(\varepsilon u,\varepsilon v\right)F_N(u,v)\,|\partial_u\psi_N\times\partial_v\psi_N|du\,dv\\
&\quad+\int_{\mathbb{R}^2}\rho_S^\varepsilon\circ\psi_S(u,v)\,F_S(u,v)\,|\partial_u\psi_S\times\partial_v\psi_S|du\,dv.
\end{aligned}$$

The last term converges to 0 as $\varepsilon\to 0$, since

- the measure of the support of $\rho_S^\varepsilon \circ \psi_S$ converges to zero;
- the integrand is uniformly bounded with respect to ε.

For the first term, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \rho\left(\varepsilon u, \varepsilon v\right) F_N\left(u, v\right) |\partial_u \psi_N \times \partial_v \psi_N| du\, dv \\
&= \int_{B_{\varepsilon^{-1}}(0)} F_N\left(u, v\right) |\partial_u \psi_N \times \partial_v \psi_N| du\, dv \\
&+ \int_{B_{2\varepsilon^{-1}}(0) \setminus B_{\varepsilon^{-1}}(0)} \rho\left(\varepsilon u, \varepsilon v\right) F_N\left(u, v\right) |\partial_u \psi_N \times \partial_v \psi_N| du\, dv.
\end{aligned}$$

The last summand of this expression converges to zero, since

- $|\rho\left(\varepsilon u, \varepsilon v\right) F_N\left(u, v\right)| \leq \sup_{S^2} |f|$;
- $\int_{B_{2\varepsilon^{-1}}(0) \setminus B_{\varepsilon^{-1}}(0)} |\partial_u \psi \times \partial_v \psi| du\, dv \leq Area\big(S^2 \setminus \psi_N(B_{\varepsilon^{-1}}\left(0\right))\big) \to 0.$

Summing up, we obtain

$$\int_{S^2} f = \int_{\mathbb{R}^2} F_N\left(u, v\right) |\partial_u \psi_N \times \partial_v \psi_N|\, du\, dv \tag{1.114}$$

just as it is well-known from the analysis course.

Of course, a similar argument yields also

$$\int_{S^2} f = \int_{\mathbb{R}^2} F_S\left(u, v\right) |\partial_u \psi_S \times \partial_v \psi_S|\, du\, dv. \tag{1.115}$$

The reader should check directly that the right hand sides of (1.114) and (1.115) are equal indeed.

Theorem 1.116. *Let $h\colon S_1 \to S_2$ be a diffeomorphism, where S_1 and S_2 are compact surfaces. Then for any $f \in C^\infty\left(S\right)$ we have*

$$\int_{S_2} f = \int_{S_1} \left(f \circ h\right) \cdot |\det dh|. \tag{1.117}$$

To explain the right hand side of (1.117), let V and W be Euclidean vector spaces such that $\dim \mathrm{V} = \dim \mathrm{W} = n$. Choose an orthonormal basis $e = (e_1, \dots, e_n)$ of V and an orthonormal basis $g = (g_1, \dots, g_n)$ of W. A linear map $\varphi\colon \mathrm{V} \to \mathrm{W}$ can be represented by a matrix $A_\varphi = \left(a_{ij}\right) \in M_n\left(\mathbb{R}\right)$, where

$$\varphi\left(e_i\right) = \sum_{j=1}^n a_{ij} g_j \quad \Longleftrightarrow \quad \left(\varphi\left(e_1\right), \dots, \varphi\left(e_n\right)\right) = \left(g_1, \dots, g_n\right) \cdot A \quad \Longleftrightarrow \quad \varphi\left(e\right) = g \cdot A.$$

If e' is another basis of V, then there exits an orthogonal $n \times n$ matrix B such that

$$e' = e \cdot B \qquad \Longleftrightarrow \qquad e_i' = \sum_{j=1}^n b_{ij} e_j.$$

Similarly, if g' is another basis of W, then there exists an orthogonal $n \times n$ matrix $C = (c_{ij})$ such that

$$g' = g \cdot C \qquad \Longleftrightarrow \qquad g'_i = \sum_{j=1}^{n} c_{ij} g_j .$$

Let A'_φ be the matrix of φ with respect to e' and g'. Then

$$\begin{aligned} \varphi\left(e'\right) = g' \cdot A'_\varphi = gCA'_\varphi \quad &\Longleftrightarrow \quad \varphi\left(e \cdot B\right) = \varphi\left(e\right) \cdot B = g \cdot A_\varphi B \\ &\Longrightarrow \quad CA'_\varphi = A_\varphi B \quad \Longrightarrow \quad A'_\varphi = C^{-1} A_\varphi B. \end{aligned}$$

Therefore,

$$\det A'_\varphi = \det\left(C^{-1}\right) \det A_\varphi \det B = \pm \det A_\varphi \quad \Longrightarrow \quad \left|\det A'_\varphi\right| \; = \left|\det A_\varphi\right|,$$

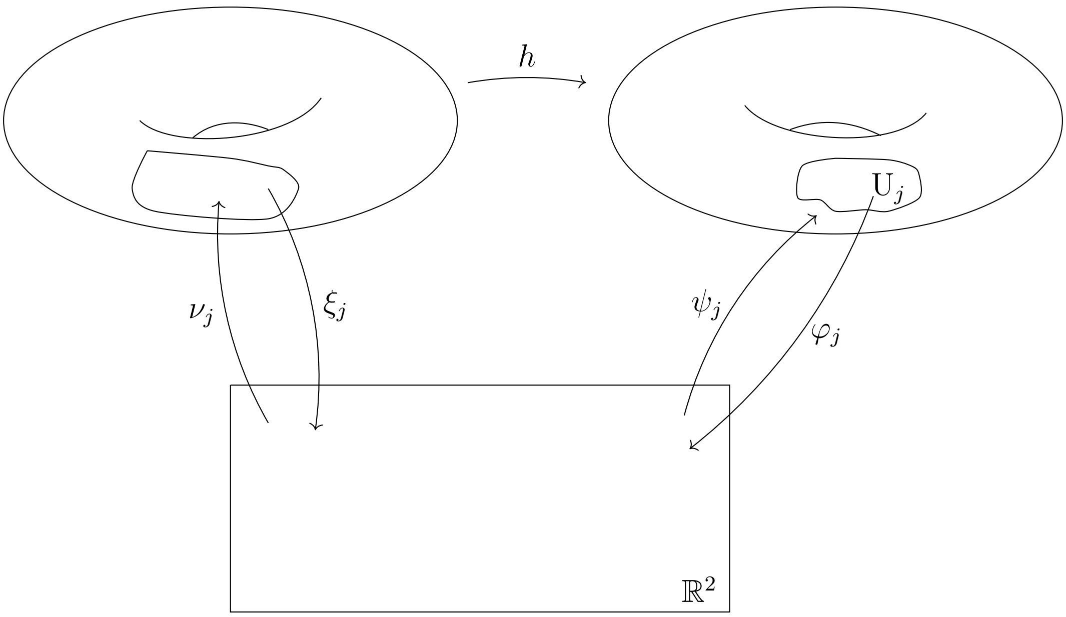
since both $\det\left(C^{-1}\right)$ and $\det B$ equal to ± 1 because B and C are orthogonal. That is for any linear map $\varphi\colon \mathrm{V} \to W$ between Euclidean spaces $\left|\det \varphi\right| := \left|\det A_\varphi\right|$ is well-defined.

Since for any $p \in S_1$ both $T_p S_1$ and $T_{h(p)} S_2$ are Euclidean, $\left|\det dh\right|$ is a well-defined function on S_1.

Proof of Theorem 1.116. Let $\mathcal{U}_2 = \{(\mathrm{U}_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ be an atlas on S_2. Pick a partition of unity $\{\rho_j \mid j = 1, \dots, n\}$ on S_2 subordinate to $\mathcal{U}_2$. Then $\mathcal{U}_1 = \{(h^{-1}(\mathrm{U}_\alpha), \xi_\alpha := \varphi_\alpha \circ h) \mid \alpha \in A\}$ is an atlas on S_1 and $\{\rho_j \circ h \mid j = 1, \dots, n\}$ is a partition of unity subordinate to $\mathcal{U}_1$. If $\operatorname{supp} \rho_j \subset \mathrm{U}_{\alpha_j} =: \mathrm{U}_j$, denote $\psi_j = \varphi_j^{-1}$, $\xi_j = \varphi_{\alpha_j} \circ h$ and $\nu_j = \xi_j^{-1} = h^{-1} \circ \psi_j$. Hence,

$$\begin{aligned} \psi_j = h \circ \nu_j \qquad &\Longrightarrow \qquad \partial_u \psi_j = dh\left(\partial_u \upsilon_j\right) \;\text{ and }\; \partial_v \psi_j = dh\left(\partial_v \upsilon_j\right) \\ &\Longrightarrow \qquad \left|\partial_u \psi_j \times \partial_v \psi_j\right| = \left|\det dh\right| \left|\partial_u \nu_j \times \partial_v \nu_j\right|. \end{aligned} \tag{1.118}$$

The last equality follows from the following fact: If $E \subset \mathbb{R}^3$ is a plane spanned by two vectors v and w, for any $A \in \operatorname{End}(E)$ we have $(Av) \times (Aw) = (\det A) \cdot v \times w$.



Thus, we have

$$\int_{S_1} (\rho_j \circ h) \cdot (f \circ h) \cdot |\det dh| = \int_{\mathbb{R}^2} \left(\rho_j \circ h \circ \xi_j^{-1}\right) \cdot \left(f \circ h \circ \xi_j^{-1}\right) \cdot |\det dh| \cdot |\partial_u \nu_j \times \partial_v \nu_j|$$
$$= \int_{\mathbb{R}^2} (\rho_j \circ \psi_j) \cdot (f \circ \psi_j) \, |\partial_u \psi_j \times \partial_v \psi_j|$$
$$= \int_{S_2} \rho_j \cdot f,$$

where the second equality follows from (1.118). Summing up by j, we obtain (1.117). $\square$

Remark 1.119. Notice that (1.117) is nothing else but a fancy restatement of the theorem about the change of coordinates for the integration, which is well-known from the analysis course.

1.9 Quadratic forms on surfaces

Definition 1.120. A Riemannian metric on a smooth surface S is a family of scalar products $\{\langle \cdot, \cdot \rangle_p \mid p \in S\}$, where $\langle \cdot, \cdot \rangle_p$ is a scalar product on T_pS, such that $\langle \cdot, \cdot \rangle_p$ depends smoothly on p.

To explain, let $\psi \colon \mathrm{V} \to \mathrm{U}$ be a parametrization. If $q \in \mathrm{V}$ and $p = \psi(q)$, then T_pS has a basis $(\partial_u \psi, \partial_v \psi)$. Hence, the scalar product $\langle \cdot, \cdot \rangle_p$ is represented by its Gram matrix

$$M = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \qquad \text{where} \qquad \begin{aligned} E &= \langle \partial_u \psi, \partial_u \psi \rangle_p, \\ F &= \langle \partial_u \psi, \partial_v \psi \rangle_p, \\ G &= \langle \partial_v \psi, \partial_v \psi \rangle_p. \end{aligned}$$

We say, that $\langle \cdot, \cdot \rangle_p$ depends smoothly on p, if all 3 functions E, F, G are smooth on U (where they are defined).

Example 1.121. For any $p \in S$ we have $T_pS \subset \mathbb{R}^3$. Since $\mathbb{R}^3$ is equipped with the standard scalar product.

$$\langle x, y \rangle_{st} := x_1 y_1 + x_2 y_2 + x_3 y_3$$

we can restrict $\langle \cdot, \cdot \rangle_{st}$ to T_pS to obtain a scalar product on T_pS. This is a Riemannian metric on S, since

$$E(u, v) = \langle \partial_u \psi, \partial_u \psi \rangle_S = \langle \partial_u \psi, \partial_u \psi \rangle_{st}$$

is a smooth function of (u, v) (and similarly for F and G).

This particular Riemannian metric on S is called *the first fundamental form* of S in the classical theory of surfaces.

Exercise 1.122. Let $\langle \cdot, \cdot \rangle$ be the first fundamental form of S and $f \colon S \to S$ be a diffeomorphism. For $v, w \in T_pS$ define a new scalar product

$$\langle v, w \rangle_f := \langle d_p f(v), d_p f(w) \rangle_{f(p)}$$

where $d_p f(v) \in T_{f(p)}S$ and $d_p f(w) \in T_{f(p)}S$. Show that $\langle \cdot, \cdot \rangle_f$ is a Riemannian metric on S.

For the sake of simplicity of exposition, assume S is oriented and let n be the unit normal field. We can regard n as a smooth map

$$n\colon S \longrightarrow S^2,$$

which is called *the Gauss map*. Then for all $p \in S$ we have

$$d_p n\colon T_pS \longrightarrow T_{n(p)}S^2 = n(p)^{\perp} = T_pS.$$

This linear map is called *the shape operator* of S at p.

As a linear map in a 2-dimensional vector space, the shape operator has two invariants:

$$K(p) := \det(d_p n) \qquad \textit{and} \qquad H(p) := -\frac{1}{2}\operatorname{tr}(d_p n).$$

Definition 1.123. $K(p)$ is called *the Gauss curvature* and $H(p)$ is called *the mean curvature* of S at p.

Notice that both K and H are smooth functions on S.

Example 1.124. For the plane $S = \mathbb{R}^2 \equiv \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ the Gauss map is constant. Hence, the shape operator vanishes and therefore both K and H vanish too.

Example 1.125. For the sphere of radius r

$$S^2_r := \left\{x \in \mathbb{R}^3 \mid |x|^2 = r^2\right\}$$

the Gauss map is given by $n(p) = \frac{1}{r}p$. Hence, for the shape operator we obtain: $d_p n(v) = \frac{1}{r}v$. Thus, $d_p n = \frac{1}{r}\mathrm{id} \;\Rightarrow\; K(p) = \frac{1}{r^2}$ is constant on S^2.

Notice that for $r \to \infty$, we have $K(p) \to 0$ and the sphere looks more and more flat in a neighbourhood of each point (that is why our Earth is "flat"). Thus, we can view the Gauss curvature as a measure of flatness of S.

Lemma 1.126. *The shape operator is symmetric, that is for any $p \in S$ and any $v, w \in T_pS$ we have*

$$\langle d_p n(v), w\rangle = \langle v, d_p n(w)\rangle.$$

Proof. Let $\psi\colon \mathrm{V} \to S$ be a parametrization such that $\psi(0) = p$. Then $(\partial_u\psi, \partial_v\psi)\Big|_{(u,v)=0}$ is a basis of T_pS. Hence, it suffices to show the equality

$$\big\langle d_p n(\partial_u\psi), \partial_v\psi\big\rangle = \big\langle \partial_u\psi, d_p n(\partial_v\psi)\big\rangle, \tag{1.127}$$

where the derivatives are evaluated at the origin. To this end, notice that by the definition of n we have

$$\big\langle n(\psi(u,v)), \partial_u\psi(u,v)\big\rangle = 0 \qquad \forall (u,v) \in \mathrm{V}.$$

Differentiating this equality with respect to v and setting $(u,v) = 0$, we obtain

$$\big\langle d_p n(\partial_u\psi), \partial_v\psi\big\rangle + \big\langle n(p), \partial_{uv}\psi\big\rangle = 0.$$

Similarly, we obtain

$$\big\langle \partial_u\psi, d_p n(\partial_v\psi)\big\rangle + \big\langle \partial_{uv}\psi, n(p)\big\rangle = 0.$$

Subtracting these two equalitites, we arrive at (1.127). $\square$

Definition 1.128. The bilinear symmetric map

$$\mathrm{II}\colon T_pS \times T_pS \longrightarrow \mathbb{R}, \qquad (v, w) \longmapsto \langle v, d_pn\,(w)\rangle_p$$

is called *the second fundamental form* of S at p.

Notice that II is smooth, that is for any parametrization ψ the functions

$$\mathrm{II}\big(\partial_u\psi\,(u,v)\,, \partial_u\psi\,(u,v)\,\big), \qquad \mathrm{II}\big(\partial_u\psi, \partial_v\psi\big), \qquad \mathrm{II}\big(\partial_v\psi, \partial_v\psi\big)$$

are smooth in (u, v).

Remark 1.129. One can recover the shape operator from the second fundamental form, that is these two objects contain the same amount of information.

Remark 1.130. Observe that the shape operator and the second fundamental form depend on the choice of orientation. If one changes the orientation to the opposite one, the sign of the fundamental form changes too. However, this does not affect the Gauss curvature. In particular, the Gauss curvature is well-defined for non-orientable surfaces too, since any surface is *locally* orientable.

1.10 The geometric meaning of the sign of the Gauss curvature

Let $p \in S$ be a critical point of $f \in C^\infty(S)$. Given $v \in T_pS$, pick a smooth curve $\gamma\colon\ (-\varepsilon, \varepsilon) \to S$ such that $\gamma\,(0) = p$ and $\dot\gamma\,(p) = v$.

Definition 1.131. The map

$$\mathrm{Hess}_p f\colon T_pS \longrightarrow \mathbb{R}, \qquad \mathrm{Hess}_p f\,(v) = \frac{d^2}{dt^2}\bigg|_{t=0} (f \circ \gamma\,(t))$$

is called the Hessian of f at p.

Proposition 1.132.

(i) $\mathrm{Hess}_p f$ *is a well-defined quadratic map.*

(ii) If p *is a point of local minimum, then* $\mathrm{Hess}_p\,(f)\,(v) \geq 0$ *for all* $v \in T_pS$. *If* p *is a point of local maximum, then* $\mathrm{Hess}_p f\,(v) \leq 0$.

(iii) If $\mathrm{Hess}_p f\,(v) > 0$ *for all* $v \neq 0$*, then* p *is a point of local minimum. If* $\mathrm{Hess}_p f\,(v) < 0$ *for all* $v \neq 0$*, then* p *is a point of local maximum.*

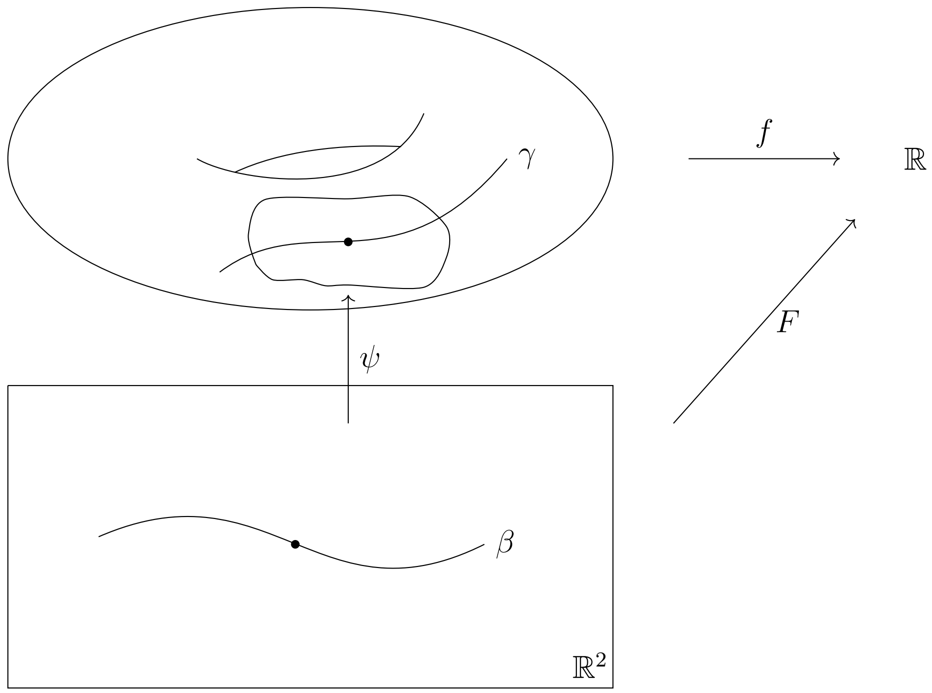
Proof. Choose a parametrization ψ such that $\psi\,(0) = p$ and denote

$$F := f \circ \psi \qquad \text{and} \qquad \beta := \varphi \circ \gamma = \psi \circ \gamma.$$

Then if $\beta\,(t) = \big(\beta_1\,(t)\,, \beta_2\,(t)\,\big)$, we have

$$f \circ \gamma\,(t) = F \circ \beta\,(t) = F\big(\beta_1\,(t)\,, \beta_2\,(t)\,\big) \quad \Longrightarrow$$
$$\frac{d}{dt} f \circ \gamma\,(t) = \partial_u F\,(\beta\,(t))\,\beta_1'\,(t) + \partial_v F\,(\beta\,(t))\,\beta_2'\,(t)\,.$$

Notice that $\beta\,(0) = 0$ and $\partial_u F\,(0) = 0 = \partial_v F\,(0)$.



Furthermore we have

$$\frac{d^2}{dt^2}\bigg|_{t=0} f \circ \gamma\left(t\right) = \partial^2_{uu} F\left(0\right) \beta_1'\left(0\right)^2 + 2\partial^2_{uv} F\left(0\right) \beta_1'\left(0\right) \beta_2'\left(0\right) + \partial^2_{vv} F\left(0\right) \beta_2'\left(0\right)^2. \qquad (1.133)$$

Recalling that $\beta'\left(0\right) = d_p\varphi\left(v\right)$, we see that the right-hand-side of (1.133) depends only on $\beta'\left(0\right)$ and not on the choice of γ. Moreover, (1.133) also shows that $\mathrm{Hess}_p f\left(v\right)$ is a quadratic form in v.

In fact the above computation shows that $\mathrm{Hess}_p f$ corresponds to the Hessian of the local representation F of f in the following sense: The diagram

$$\begin{tikzcd}
T_pS \arrow[dr, "\mathrm{Hess}_p f"] \arrow[dd, "d_p\varphi"'] & \\
& \mathbb{R} \\
\mathbb{R}^2 \arrow[ur, "\mathrm{Hess}_{\varphi(p)} F"'] &
\end{tikzcd}$$

commutes. That is we can identify $\mathrm{Hess}_p f$ with $\mathrm{Hess}_{\varphi(p)} F$ by means of the isomorphism $d_p\varphi \colon T_pS \to \mathbb{R}^2$. This immediately implies *(ii)* and *(iii)*. □

Let $H_a \colon \mathbb{R}^3 \to \mathbb{R}$, $H_a(x) = \langle x, a\rangle$, be the height function in the direction of $a \in \mathbb{R}^3$, $a \neq 0$. Denote by h_a the restriction of H_a to S, see Example 1.26. Recall that p is a critical point of h_a if and only if $T_pS \perp a$. For example, for $a = (0, 0, 1)$ we have the standard height function, which has 4 critical points on the torus as shown on Figure 1.21 below.

Proposition 1.134. *Let n be an orientation of S. Then for any $p \in S$ we have*

$$\mathrm{II}_p = -\mathrm{Hess}_p\left(h_{n(p)}\right).$$

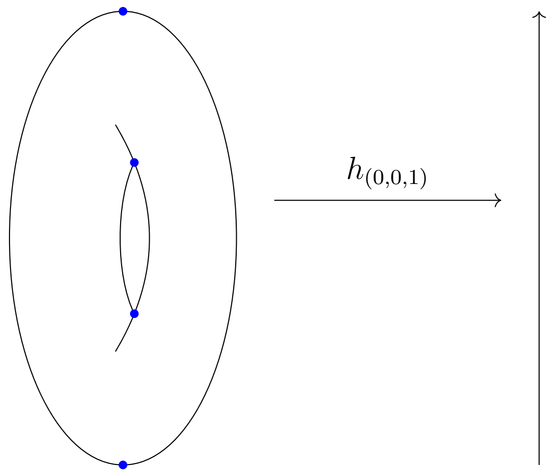


Figure 1.21: Critical points of the standard height function in the torus

Proof. Observe first that $T_pS \perp n(p)$ implies that p is a critical point of $h_{n(p)}$.

Given $v \in T_pS$, choose a curve $\gamma\colon (-\varepsilon, \varepsilon) \to S$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. Then

$$\mathrm{Hess}_p\left(h_{n(p)}\right) = \frac{d^2}{dt^2}\bigg|_{t=0} \langle \gamma(t), n(p)\rangle = \langle \ddot\gamma(0), n(p)\rangle .$$

However,

$$\gamma(t) \in S \qquad \Longrightarrow \qquad \dot\gamma(t) \in T_{\gamma(t)}S \quad \forall t \qquad \Longrightarrow \qquad \langle \dot\gamma(\gamma), n(\gamma(t))\rangle = 0 \qquad \forall t.$$

Differentiating the last equality in t, we obtain

$$\langle \ddot\gamma(0), n(p)\rangle + \langle \dot\gamma(0), d_pn(\dot\gamma(0))\rangle = 0.$$

where the second summand equals $\mathrm{II}_p(v)$. This yields

$$\mathrm{II}_p(v) = -\langle \ddot\gamma(0), n(p)\rangle = -\mathrm{Hess}_p\left(h_{n(p)}\right) .$$

□

Fix $p \in S$. Without loss of generality assume that

$$p = 0 \in \mathbb{R}^3 \qquad \textit{and} \qquad n(0) = (0,0,1) .$$

This can be always achieved by applying a translation and a rotation in $\mathbb{R}^3$.

Since the shape operator $d_0n\colon T_0S \to T_0S$, where $T_0S = \mathbb{R}^2$, is symmetric, d_0n has two real eigenvalues, say k_1 and k_2. Consider the following cases:

A) If $K(p) > 0$, then $\det(d_pn) = k_1 \cdot k_2 > 0$ so that k_1 and k_2 are either both positive or both negative. Hence, $\mathrm{Hess}_0\left(h_{n(0)}\right)$ is either positive-definite or negative definite, that is the height function in the direction $(0,0,1)$, which is simply $z\big|_S$, has a local minimum or local maximum at $p = 0$. Hence, there exists a neighbourhood U of p in S such that U lies either above or below T_pS as shown on Figure 1.22 below.

B) If $K(p) < 0$, then $z|_S$ attains both positive and negative values on each neighbourhood of p. In other words, in any neighbourhood of p there are points in S above and below T_pS as shown on Figure 1.23 below.

Remark 1.135. If $K(p) = 0$, in general one cannot say anything about the position of S relative to T_pS.

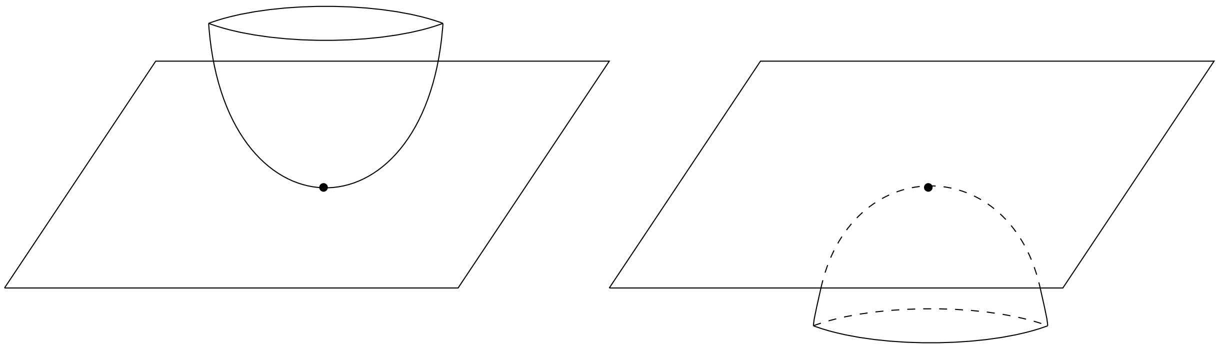


Figure 1.22: Local shape of a surface with positive Gauss curvature

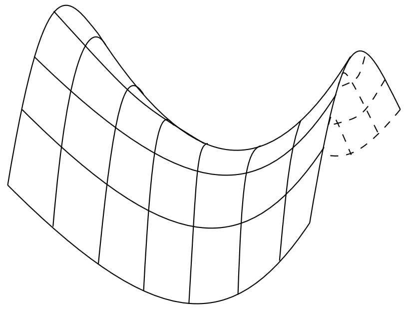


Figure 1.23: A local shape of a surface with negative curvature: the saddle surface

1.11 Surfaces of positive curvature and the Gauss–Bonnet theorem

Throughout this section, assume that S is a smooth connected surface.

Theorem 1.136 (Jordan separation theorem)**.** *If S is closed as a subset of $\mathbb{R}^3$, then $\mathbb{R}^3\backslash S$ has exactly two connected components, whose common boundary is S.* $\square$

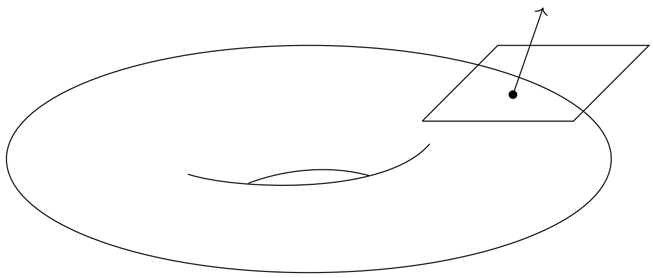
Remark 1.137. The Jordan separation theorem is a well-known result from topology. However its proof relies on certain results, which are typically not proved in a standard course in topology. Hence, we take the Jordan separation theorem as granted. An interested reader may find a proof in [MR09, Thm. 4.16].

If S is compact, then one and only one component of $\mathbb{R}^3\backslash S$ is bounded. This bounded open domain is called *the inner domain of S*. The unbounded domain is called *the outer domain of S*.

Corollary 1.138. *Any compact surface in $\mathbb{R}^3$ is orientable.*

Proof. Let $S \subset \mathbb{R}^3$ be a compact surface. Without loss of generality we can assume that S is connected (otherwise, pick a connected component of S).

Pick a point $p \in S$. A unit vector n, which is normal at p, is said to be *pointing* outwards, if there exists $\varepsilon > 0$ such that $p + tn \in \Omega_{out}$ for all $t \in (0, \varepsilon)$, where Ω_{out} is the outer domain of S.



Pick a neighbourhood W of p in $\mathbb{R}^3$ and a smooth function $\varphi\colon W \to \mathbb{R}$ such that $S \cap W = \varphi^{-1}(0)$ and $\nabla\varphi(x) \neq 0$ for all $x \in W$.

Exercise 1.139. Show that $\varphi\Big|_{\Omega_{in}\cap W} < 0$ and $\varphi\Big|_{\Omega_{out}\cap W} > 0$ (or the other way around). In other words,

$$\Omega_{in} \cap W = \{\varphi < 0\} \quad \textit{and} \quad \Omega_{out} \cap W = \{\varphi > 0\},$$

which we assume for the sake of definiteness.

Since

$$\varphi\big(p + t\nabla\varphi(p)\big) = \varphi(p) + |\nabla\varphi(p)|^2 \cdot t + O(t^2) = 0 + |\nabla\varphi(p)|^2 \cdot t + O(t^2) > 0$$

provided $t > 0$ is sufficiently small, we obtain that $\frac{\nabla\varphi(p)}{|\nabla\varphi(p)|}$ is pointing outwards for any $p \in S \cap W$. A similar argument shows that $-\frac{\nabla\varphi(p)}{|\nabla\varphi(p)|}$ is pointing inwards.

Let $\widehat{W}$ be any other open subset of $\mathbb{R}^3$ and $\widehat{\varphi} \in C^\infty\big(\widehat{W}\big)$ such that

$$S \cap \widehat{W} = \widehat{\varphi}^{-1}(0), \qquad \nabla\widehat{\varphi}(x) \neq 0 \qquad \forall x \in \widehat{W},$$
$$\Omega_{in} \cap \widehat{W} = \big\{\widehat{\varphi} < 0\big\} \quad \text{and} \quad \Omega_{out} \cap \widehat{W} = \big\{\widehat{\varphi} > 0\big\}.$$

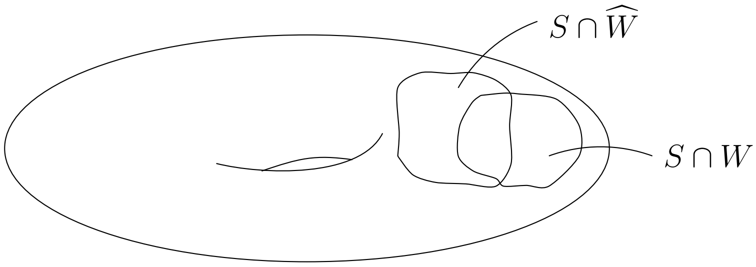
Then $\frac{\nabla\widehat{\varphi}(p)}{|\nabla\widehat{\varphi}(p)|}$ is necessarily pointing outwards. In particular,

$$\frac{\nabla\widehat{\varphi}(p)}{|\nabla\widehat{\varphi}(p)|} = \frac{\nabla\varphi(p)}{|\nabla\varphi(p)|} \qquad \forall p \in W \cap \widehat{W} \cap S.$$

That is

$$n(p) := \begin{cases} \frac{\nabla\varphi(p)}{|\nabla\varphi(p)|} & \textit{if } p \in S \cap W, \\ \frac{\nabla\widehat{\varphi}(p)}{|\nabla\widehat{\varphi}(p)|} & \textit{if } p \in S \cap \widehat{W}, \end{cases}$$

is well-defined and smooth on $S \cap \big(W \cup \widehat{W}\big)$.



Since we can cover all of S by such subsets, n is a well-defined unit normal field pointing outwards. □

Corollary 1.140. *Let S be a compact surface with positive Gauss curvature. If n is the unit normal field pointing outwards, then the second fundamental form of S with respect to n is positive-definite.*

Proof. Notice first that the second fundamental form of S with respect to the outwards pointing normal field is either positive-definite or negative-definite everywhere on S, because S is connected. Thus, it suffices to find a single point p on S such that II_p is positive-definite.

Since S is compact, there exists some $R > 0$ such that the closed ball $\bar{B}_R(0)$ of radius R centered at the origin contains S. Decreasing R if necessary, we can find a new number $R > 0$, still denoted by the same letter, such that $S \subset \bar{B}_R(0)$ and $S \cap \partial\bar{B}_R(0) \neq \varnothing$.

Let p be a point in $S \cap \partial\bar{B}_R(0)$ and γ any curve on S through p. Clearly, the function $t \mapsto \|\gamma(t)\|^2$ has a maximum at $t = 0$ which yields

$$0 = d\frac{d}{dt}\Big|_{t=0}\|\gamma(t)\|^2 = 2\big\langle\dot{\gamma}(0),\, p\big\rangle.$$

Hence, $T_pS = p^{\perp}$. Moreover, the vector p is pointing outwards, hence $n(p) = p/|p|$. Now it is clear that the height function $h_{n(p)}$ has a maximum at p and therefore

$$\mathrm{Hess}\big(h_{n(p)}\big) = -\mathrm{II}_p < 0 \qquad \Longleftrightarrow \qquad \mathrm{II}_p > 0.$$

$\square$

Proposition 1.141. *Let $S \subset \mathbb{R}^3$ be a compact connected surface. If $K(p) > 0$ for all $p \in S$, then Ω_{in} is convex, that is*

$$x, y \in \Omega_{in} \qquad \Longrightarrow \qquad [x, y] \subset \Omega_{in},$$

where $[x, y]$ is the segment in $\mathbb{R}^3$ connecting x and y. In particular, $\overline{\Omega}_{in}$ is also convex and, moreover, if $x, y \in S$, then $]x, y[\, \subset \Omega_{in}$.

Proof. Assume $\Omega = \Omega_{in}$ is not convex. Consider $A := \{(x, y) \in \Omega \times \Omega \mid [x, y] \subset \Omega\}$.

Notice that

- $A \neq \varnothing$, since $(x, x) \in A$ for all $x \in \Omega$;
- $A \neq \Omega \times \Omega$, since otherwise Ω were convex.

Then the topological boundary ∂A of $A \subset \Omega \times \Omega$ is non-empty. This means the following: there exist sequences $x_n, y_n, x'_n, y'_n \in \Omega$ such that

$$x_n, x'_n \longrightarrow x \in \Omega, \quad \text{and} \quad y_n, y'_n \longrightarrow y \in \Omega \quad \text{such that}$$
$$[x_n, y_n] \subset \Omega \quad \text{and} \quad [x'_n, y'_n] \not\subset \Omega.$$

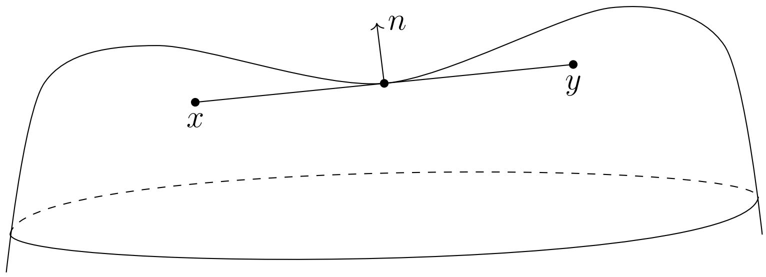
Exercise 1.142. Show that there exists $z \in [x, y] \cap \partial\Omega$, where $\partial\Omega = S$, such that $v := y - x \in T_zS$. In particular, $[x, y] \subset T_zS$.

Assuming Exercise 1.142, we proceed as follows. Let n be a unit normal vector at z pointing outwards (locally, so that a neighbourhood of z in S is located below the tangent plane). Then $\mathrm{Hess}_z h_n < 0$ so that h_n has a strict local maximum at z. Furthermore, we can assume that

$$z = 0, \quad n = (0, 0, 1), \quad v = (1, 0, 0), \quad \text{and} \quad S = \{(u, v, f(u, v))\}$$

in a neighbourhood of the origin.

Consider the curve $\gamma\colon (-\varepsilon, \varepsilon) \to S$ given by $\gamma(t) = \big(t, 0, f(t, 0)\big)$. Since $\gamma(t)$ lies above $(t, 0, 0)$, we must have $f(t, 0) \geq 0$ and $f(0, 0) = 0$. Hence, $t = 0$ must be a point of local minimum for the function $t \mapsto f(t, 0)$. However, this is impossible, because $h_n \circ \gamma\colon t \mapsto f(t, 0)$ must have a strict local maximum at $t = 0$. $\square$



Proposition 1.143. *Let S be a surface with positive Gauss curvature. The affine tangent plane*

$$T_p^a S = \{p + v \mid v \in T_p S\}$$

intersects S at p only.

Proof. Assume $q \in T_p^a S \cap S$ and $q \neq p$. Then $]p, q[\, \in \Omega_{in}$ by Proposition 1.141. However, the positivity of the Gauss curvature implies that all points in a neighbourhood of p in $T_p^a S$ lie in Ω_{out}. This is a contradiction. □

Theorem 1.144. *Let S be a compact connected surface. If $K(p) > 0$ for all $p \in S$, then the Gauss map $n\colon S \to S^2$ of S is a diffeomorphism.*

Proof. The proof of this theorem consists of the following steps.

Step 1. *The Gauss map is a local diffeomorphism.*

Since $K(p) := \det(d_p n) \neq 0$, $d_p n$ is an isomorphism. Hence, n is a local diffeomorphism by Theorem 1.70.

Step 2. *The Gauss map is surjective.*

Since S is compact and n is continuous, $n(S) \subset S^2$ is a compact subset. Therefore, $n(S)$ is closed, because S^2 is Hausdorff. Also, $n(S)$ is clearly non-empty.

Furthermore, Step 1 implies that $n(S)$ is open. Since S^2 is connected, we must have $n(S) = S^2$.

Step 3. *The Gauss map is injective.*

Given $n \in S^2$ consider the height function

$$H_n\colon \overline{\Omega}_{in} \longrightarrow \mathbb{R}, \qquad x \longmapsto \langle n, x \rangle$$

so that $H_n\Big|_{\partial\overline{\Omega}_{in} = S} = h_n$. Notice that any point of local maximum of H_n must be on $\partial\overline{\Omega}_{in} = S$, since $\nabla H_n \neq 0$ at any interior point of $\overline{\Omega}_{in}$.

Assume H_n has two distinct points of local maxima. Denote these points by p and q. Without loss of generality we can assume $H_n(p) \geq H_n(q)$. It is convenient to consider the following two cases separately.

Case 1. $H_n(p) > H_n(q)$.

In this case for any $t > 0$ we have

$$\begin{aligned} H_n(tp + (1-t)q) &= tH_n(p) + (1-t)H_n(q) \\ &> tH_n(q) + (1-t)H_n(q) = H_n(q). \end{aligned}$$

For $t > 0$ and $t \to 0$, we have $p_t := tp + (1-t)q \to q$ and $H_n(p_t) > H_n(q)$. Thus, q cannot be a point of local maximum for H_n.

Case 2. $H_n(p) = H_n(q)$.

We have

$$\begin{aligned} H_n(p) = H_n(q) &\iff \langle n, p - q \rangle = 0 \\ &\implies p - q \in T_pS \\ &\implies p + t(p-q) \in T_p^aS \qquad \forall t \in \mathbb{R} \\ &\overset{t=-1}{\implies} q \in T_p^aS \\ &\implies q = p. \end{aligned}$$

However, this contradicts the assumption that p and q are distinct.

Thus, H_n has at most one local maximum on $\overline{\Omega}_{in}$. Since $\overline{\Omega}_{in}$ is compact, such point must exist, so that H_n has a unique point of local maximum p, which lies on S. Then p is also a unique point of local maximum for h_n, that is a unique solution of $n(q) = n$. This finishes the proof of Step 3.

The proof of this theorem now follows easily from the preceding steps. Indeed, Steps 2 and 3 yield that the inverse of the Gauss map exits and Step 1 immediately implies that n^{-1} is smooth. □

Corollary 1.145. *Let S be any compact surface with positive Gauss curvature K. Then*

$$\int_S K = 4\pi. \tag{1.146}$$

Proof. The claim of this corollary follows from the following computation

$$\int_S K = \int_S |K| = \int_S |\det(dn)| = \int_{S^2} 1 = \text{Area}\left(S^2\right) = 4\pi,$$

where the first equality follows from $K > 0$, the second one from the definition of K, and the third one from Theorem 1.116. □

Remark 1.147. It turns out that albeit we did use the hypothesis $K > 0$ in the proof, (1.146) still holds for any S diffeomorphic to S^2.

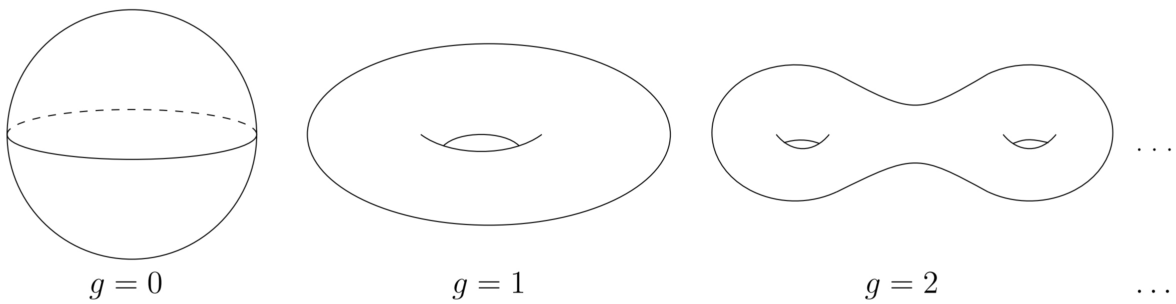


Figure 1.24: Surfaces with 0 holes (the sphere), 1 hole (the torus), and 2 holes

Even more generally, let g denote the number of "holes" of S as shown on Fig. 1.24. Then we have

$$\int_S K = 4\pi(1 - g)$$

provided S is compact and orientable. This is the celebrated Gauss-Bonnet theorem.

1.11.1 A solution of Exercise 1.142

Since $[x'_n, y'_n] \not\subset \Omega$, there exists some $t_n \in [0,1]$ such that $z'_n = t_n x'_n + (1-t_n)\, y'_n \notin \Omega$. By the compactness of $[0,1]$, there exists a subequence t_{n_m} converging to some $t \in [0,1]$. In fact, $t \in (0,1)$ since the endpoint of $[x,y]$ belong to Ω by construction.

Furthermore, any neighbourhood of $z := tx + (1-t)\, y$ contains points from the complement of Ω, for example z'_{n_m} for m sufficiently large. However, any neighbourhood of z contains also points from Ω, for example $z_{n_m} := t_{n_m} x_{n_m} + (1-t_{n_m})\, y_{n_m}$ provided m is sufficiently large. Hence, $z \in \partial\Omega = S$.

Assume $v \notin T_z S$. Then any neighbourhood of z in $[x,y]$ would contain points both from Ω and $\mathbb{R}^3 \backslash \Omega$. Indeed, if S is given by the equation $\varphi(p) = 0$ in a neighbourhood of z, then

$$v \notin T_z S \quad \Longleftrightarrow \quad \langle \nabla\varphi(z), v\rangle \neq 0 \quad \Longrightarrow \quad \varphi(z + tv) = 0 + t\langle \nabla\varphi(z), v\rangle + O\left(t^2\right).$$

Hence, since $\nabla\varphi(z) \neq 0$, φ takes both positive and negative values on $[z - \varepsilon v, z + \varepsilon v]$. This is impossible, since otherwise $[x_{n_m}, y_{n_m}]$ cannot be contained in Ω.

Chapter 2

Manifolds

There is a number of ways we could generalize our discussion of surfaces.

Hypersurfaces. These are subsets $S \subset \mathbb{R}^{k+1}$ admitting (smooth) parametrizations $\psi\colon \mathrm{V} \to \mathrm{U} \subset S, \mathrm{V} \subset \mathbb{R}^k$ in a neighbourhood of each point just as in the definition of the surface. Virtually all notions and theorems about surfaces we have seen above generalize immediately to this case (together with proofs), since the condition $k = 2$ was never used in an essential way.

Embedded submanifolds. Roughly speaking these are "k-dimensional surfaces in $\mathbb{R}^{k+l}$". More formally, we could call $S \subset \mathbb{R}^{k+l}$ a k-dimensional submanifold, if S admits parametrizations $\psi\colon \mathrm{V} \to \mathrm{U} \subset S$, where $\mathrm{V} \subset \mathbb{R}^k$ is open and $D\psi$ is injective at each point. Most of the statements about surfaces we have seen above generalize to this case too (and rather trivially) except the very last section involving the Gauss map. It generalizes too, however, this requires some extra work and, more importantly, not all statements made for surfaces hold true in this case.

An interested reader may find further details for example in [Tho79] or [GP10]. However, even higher degree of abstraction is required for applications. Therefore, we consider below what is known as *abstract manifolds* skipping the above intermediate steps. Abstract manifolds are basically “surfaces”, which are not necessarily contained in any ambient Euclidean space.

2.1 Abstract manifolds

Definition 2.1. A Hausdorff topological space M is said to be a topological manifold of dimension $k \in \mathbb{N}_0$, if M is locally homeomorphic to $\mathbb{R}^k$.

To explain: for all $m \in M$ there exists a neighbourhood $\mathrm{U} \subset M$ and a homeomorphism $\varphi\colon \mathrm{U} \to \mathrm{V} \subset \mathbb{R}^k$, where V is open. A pair (U, φ) is called *a chart* on M.

Example 2.2. Any surface is a topological manifold of dimension $k = 2$.

Example 2.3. S^k is a topological manifold of dimension k. This can be seen by covering S^k by two charts just in the case $k = 2$.

Example 2.4 (A non-example)**.** The union of two intersecting lines, which can be described more explicitly as $\{(x, y) \in \mathbb{R}^2 \mid x^2 = y^2\}$, is not a topological manifold. This should be compared with Examples 1.12 and 1.18.

Definition 2.5 (Smooth manifold)**.**

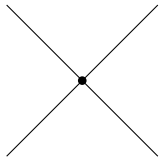


Figure 2.1: The union of two intersecting lines is not a manifold

- A collection $\mathcal{U} = \{(\mathrm{U}_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ of charts on M is called a C^0-atlas, if

$$\bigcup_{\alpha \in A} \mathrm{U}_\alpha = M.$$

- A C^0-atlas is called smooth, if each coordinate transformation

$$\theta_{\alpha\beta} \coloneqq \varphi_\alpha \circ \varphi_\beta^{-1} \colon \varphi_\beta\left(\mathrm{U}_\alpha \cap \mathrm{U}_\beta\right) \longrightarrow \varphi_\alpha\left(\mathrm{U}_\alpha \cap \mathrm{U}_\beta\right)$$

is smooth, where $\varphi_\beta\left(\mathrm{U}_\alpha \cap \mathrm{U}_\beta\right) \subset \mathbb{R}^k$ and $\varphi_\alpha\left(\mathrm{U}_\alpha \cap \mathrm{U}_\beta\right) \subset \mathbb{R}^k$.

- A smooth manifold is a topological manifold equipped with a smooth atlas.

Example 2.6.

1) $\mathbb{R}^k$ with a single chart $\left(\mathbb{R}^k, \mathrm{id}\right)$ is a smooth manifold. More generally, any open subset of $\mathbb{R}^k$ is a smooth manifold.
2) Any smooth surface is a smooth manifold of dimension $n = 2$.
3) S^k is a smooth manifold of dimension k.
4) The real projective space is defined as follows:

$$\begin{aligned}
\mathbb{RP}^k &= \text{ the set of all lines in } \mathbb{R}^{k+1} \text{ through the origin;} \\
&= \mathbb{R}^{k+1} \setminus \{0\} /\!\sim, \qquad (x_0, \ldots, x_k) \sim (\lambda x_0, \ldots, \lambda x_k) \ \text{ for some } \lambda \in \mathbb{R} \setminus \{0\}\,; \\
&= S^k / \sim, \qquad x \sim -x.
\end{aligned}$$

Define the topology on $\mathbb{RP}^k$ as the quotient topology of $\mathbb{R}^{k+1} \setminus \{0\}$, that is $\mathrm{U} \subset \mathbb{RP}^k$ is declared to be open if and only if $\pi^{-1}(\mathrm{U}) \subset \mathbb{R}^{k+1} \setminus \{0\}$ is open, where $\pi \colon \mathbb{R}^{k+1} \setminus \{0\} \to \mathbb{RP}^k$ is the quotient map.

For example, the reader should be able to show following Fig. 2.2 that $\mathbb{RP}^1$ is homeomorphic to S^1.

Define

$$\mathrm{U}_j \coloneqq \left\{[x_0 \colon \ \ldots : x_n] \in \mathbb{RP}^k \mid x_j \neq 0\right\} \quad j = 0, \ldots, k.$$

U_j is open, since

$$\pi^{-1}\left(\mathrm{U}_j\right) = \left\{(x_0, \ldots, x_k) \in \mathbb{R}^{k+1} \setminus \{0\} \mid x_j \neq 0\right\}$$

is open. Furthermore, consider the map $\varphi_j \colon \mathrm{U}_j \to \mathbb{R}^k$

$$\varphi_j\left([x]\right) = \left(\frac{x_0}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_k}{x_j}\right).$$

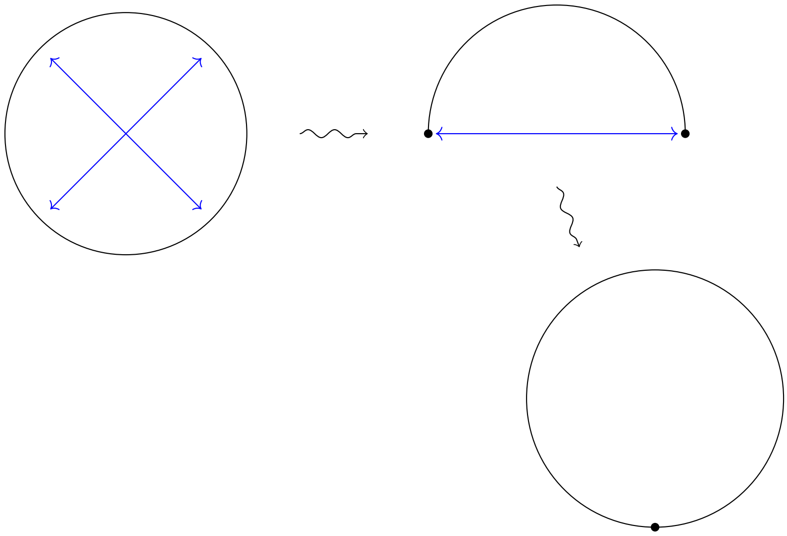


Figure 2.2: A homeomorphism between $\mathbb{RP}^1$ and S^1

Exercise 2.7. Show that φ_j is continuous. (*Hint: If X is any topological space, a map $f\colon \mathbb{RP}^k \to X$ is continuous if and only if $f \circ \pi$ is continuous.*)

Then

$$\psi_j\colon \mathbb{R}^n \to \mathrm{U}_j, \qquad \psi_j\left(y_0, \dots, y_{k-1}\right) = [y_0\colon \dots\colon y_{j-1}\colon 1\colon y_j\colon \dots\colon y_{k-1}]$$

is the inverse of φ_j. In particular, φ_j is a homeomorphism. Thus,

$$\mathcal{U} = \{(\mathrm{U}_j, \varphi_j) \mid j = 0, \dots, k\}$$

is a C^0-atlas on $\mathbb{RP}^k$.

Consider the coordinate transformation $\theta_{01} = \varphi_0 \circ \varphi_1^{-1} = \varphi_0 \circ \psi_1$, which is given by

$$\theta_{01}\left(y_0, \dots, y_{k-1}\right) = \varphi_0\left([y_0\colon 1\colon y_1\colon \dots\colon y_{k-1}]\right) = \left(\frac{1}{y_0}, \frac{y_1}{y_0}, \frac{y_2}{y_0}, \dots, \frac{y_{k-1}}{y_0}\right)$$

and is smooth on $\left\{y \in \mathbb{R}^k \mid y_0 \neq 0\right\} = \varphi_0\left(\mathrm{U}_0 \cap \mathrm{U}_1\right)$. A similar argument yields that each $\theta_{ij} = \varphi_i \circ \varphi_j^{-1}$ is smooth. Thus, $\mathcal{U}$ is in fact a smooth atlas.

Remark 2.8. For $k = 2$ we obtain a smooth manifold of dimension 2, however it turns out that $\mathbb{RP}^2$ cannot be represenetd as a surface in $\mathbb{R}^3$. We would have discovered this manifold if we would consider non-orientable surfaces more carefully. Indeed, the Gauss map of a non-orientable surface $S \subset \mathbb{R}^3$ is naturally defined as a map

$$S \ni p \longmapsto (T_pS)^{\perp} \in \mathbb{RP}^2.$$

5) Products: If M and N are smooth manifolds of dimensions k and l respectively, then $M \times N$ is a smooth manifold of dimension $k + l$. Indeed, if $\mathcal{U} = \{(\mathrm{U}_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ is a smooth atlas on M and $\mathcal{V} = \{(\mathrm{V}_\beta, \xi_\beta) \mid \beta \in B\}$ is a smooth atlas on N, that

$$\mathcal{W} := \left\{\, (\mathrm{U}_\alpha \times \mathrm{V}_\beta, \varphi_\alpha \times \xi_\beta) \mid \alpha \in A,\ \beta \in B\right\}$$

is a smooth atlas on $M \times N$.

Exercise 2.9. Find the coordinate transformations for the atlas $\mathcal{W}$ and show that these are smooth indeed.

In particular,

(i) $\mathbb{T}^k = S^1 \times \ldots \times S^1$ is a smooth manifold of dimension k;
(ii) the cylinder $\mathbb{R} \times S^1$ is a smooth manifold of dimension 2.

A smooth atlas does not need to be unique. For example, on S^2 we can choose

$$\mathcal{U} = \left\{ \left(S^2 \backslash \{N\}, \varphi_N\right), \left(S^2 \backslash \{S\}, \varphi_S\right) \right\}$$

or the atlas consisting of 6 hemispheres. However, we have seen that the resulting notions (e.g. the space of smooth functions on S^2) do not depend on this choice. The crucial point is that the charts of these two atlases are smoothly compatible, that is the corresponding coordinate transformations are smooth. This motivates to the following definition.

Definition 2.10. Two atlases $\mathcal{U} = \{(\mathrm{U}_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ and $\mathcal{V} = \{(\mathrm{V}_\beta, \xi_\beta) \mid \beta \in B\}$ on the same topological space M are said to be equivalent if $\mathcal{U} \cup \mathcal{V}$ is also a smooth atlas, that is if

$$\xi_\beta \circ \varphi_\alpha^{-1} \qquad \textit{and} \qquad \varphi_\alpha \circ \xi_\beta^{-1}$$

are smooth for all $\alpha \in A$ and for all $\beta \in B$.

Definition 2.11. *A smooth structure* is an equivalence class of atlasses.

In the sequel we shall feel free to replace an atlas by an equivalent one.

2.2 Smooth maps

Let $(M, \mathcal{U})$ be a smooth manifold.

Definition 2.12. A function $f\colon M \to \mathbb{R}$ is called smooth, if for all $\alpha \in A$ the coordinate representation

$$F_\alpha := f \circ \varphi_\alpha^{-1} \colon \mathbb{R}^k \longrightarrow \mathbb{R}$$

with respect to $(\mathrm{U}_\alpha, \varphi_\alpha)$ is smooth.

Just as in the case of surfaces each F_α is defined on an open subset of $\mathbb{R}^k$, namely $\varphi_\alpha(\mathrm{U}_\alpha)$. This should be clear by now and will not be mentioned explicitly below unless really necessary.

Exercise 2.13. Let $f\colon M \to \mathbb{R}$ be a function. If $\mathcal{U} \sim \mathcal{V}$, show that f is smooth with respect to $\mathcal{U}$ if and only if f is smooth with respect to $\mathcal{V}$.

Proposition 2.14. *The set $C^\infty(M)$ of all smooth functions on a smooth manifold is an algebra, that is*

$$\left.\begin{array}{l} f, g \in C^\infty(M) \\ \lambda, \mu \in \mathbb{R} \end{array}\right\} \qquad \Longrightarrow \qquad \lambda f + \mu g \in C^\infty(M);$$

$$f, g \in C^\infty(M) \qquad \Longrightarrow \qquad f \cdot g \in C^\infty(M).$$

The proof of this proposition is similar to the proof of Proposition 1.28 and is left as an exercise to the reader.

More generally, let $(M, \mathcal{U})$ and $(N, \mathcal{V})$ be two smooth manifolds of dimensions k and l respectively.

Definition 2.15. A map $f\colon M \to N$ is said to be smooth, if each coordinate representation

$$\xi_\beta \circ f \circ \varphi_\alpha^{-1}\colon \mathbb{R}^k \longrightarrow \mathbb{R}^l$$

is smooth.

Proposition 2.16.

$$\left.\begin{aligned} f \in C^\infty(M;N) \\ g \in C^\infty(N;L) \end{aligned}\right\} \qquad \Longrightarrow \qquad g \circ f \in C^\infty(M;L).$$

Again, the proof of this proposition is a verbatim repetition of the proof of Theorem 1.39.

Also, just in the case of surfaces, we have the notions of a diffeomorphism and a local diffeomorphism.

2.3 The tangent space

If M is an abstract manifold and $\gamma\colon (-\varepsilon, \varepsilon) \to M$ is a smooth curve, then $\dot\gamma(0)$ does not make sense in any obvious way. Hence, our definition of the tangent space does not immediately generalize to the present setting.

To come up with a suitable generalization, observe the following: $v \in \mathbb{R}^k$ is the tangent vector of a curve $\gamma\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^k$, $\gamma(0) = p$, if and only if

$$\gamma(t) = p + v \cdot t + o(t) \qquad \text{as } \ t \longrightarrow 0.$$

Hence we may consider the following equivalence relation: two smooth curves $\gamma_1, \gamma_2\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^k$ such that $\gamma_1(0) = p = \gamma_2(0)$, are said to be *equivalent* if $\gamma_1(t) - \gamma_2(t) = o(t)$.

Our observation above yields immediately the following.

Proposition 2.17. $\gamma_1 \sim \gamma_2 \iff \dot\gamma_1(0) = \dot\gamma_2(0)$. □

Hence, we may identify $\mathbb{R}^n$, which is thought of as a tangent space of $\mathbb{R}^n$ at p, with the set of equivalence classes of curves through p. Explicitly, the map $\gamma \mapsto \dot\gamma(0)$ induces a bijection

$$\{\gamma \mid \gamma(0) = p\}/\!\sim\, \longrightarrow \mathbb{R}^n.$$

Thus, we may think of tangent vectors at a given point p as classes of curves through p. This approach generalized to manifolds as follows.

Definition 2.18. Let M be a smooth manifold of dimension k. Pick a point $m \in M$. Two smooth curves $\gamma_1, \gamma_2\colon (-\varepsilon, \varepsilon) \to M$ such that $\gamma_1(0) = m = \gamma_2(0)$ are said to be equivalent, if for any chart (U, φ) such that $m \in \mathrm{U}$ we have

$$\varphi \circ \gamma_1 \sim_{\mathbb{R}^k} \varphi \circ \gamma_2 \qquad \Longleftrightarrow \qquad \varphi \circ \gamma_1(t) - \varphi \circ \gamma_2(t) = o(t). \tag{2.19}$$

An equivalence class of curves is called a tangent vector at the point m.

Lemma 2.20. *If* (2.19) *holds for* some *chart* (U, φ) *containing* m*, then* (2.19) *holds for* any *chart containing* m.

Proof. Let $\left(\widehat{\mathrm{U}}, \widehat{\varphi}\right)$ be any other chart such that $m \in \widehat{\mathrm{U}}$. Denote $p := \varphi(m)$. Then

$$\begin{aligned} \widehat{\varphi} \circ \gamma_1(t) - \widehat{\varphi} \circ \gamma_2(t) &= \underbrace{\widehat{\varphi} \circ \varphi^{-1}}_{\theta} \circ \underbrace{\varphi \circ \gamma_1}_{\beta_1}(t) - \underbrace{\widehat{\varphi} \circ \varphi^{-1}}_{\theta} \circ \underbrace{\varphi \circ \gamma_2}_{\beta_2}(t) \\ &= \theta \circ \beta_1(t) - \theta \circ \beta_2(t) \end{aligned}$$

Since θ is smooth, θ is Lipchitz, that is there exists $L > 0$ such that

$$|\theta(x) - \theta(y)| \leq L|x - y| \qquad \forall\, x, y \in B_\delta(p),$$

where $\delta > 0$. Hence,

$$|\theta \circ \beta_1(t) - \theta \circ \beta_2(t)| \leq L|\beta_1(t) - \beta_2(t)| = o(t),$$

because $\beta_1(t) - \beta_2(t) = o(t)$. $\square$

Definition 2.21. The set $T_mM := \left\{ [\gamma] \mid \gamma \text{ is a smooth curve through } m \right\}$ is called the tangent space of M at m.

Exercise 2.22. Let V be a vector space and $\Phi\colon X \to V$ a bijective map, where X is an arbitrary set. Then there is a unique structure of a vector space on X such that Φ is an isomorphism. In fact, we have

$$\lambda \cdot x = \Phi^{-1}\big(\lambda \cdot \Phi(x)\big) \qquad \text{and} \qquad x_1 + x_2 = \Phi^{-1}\big(\Phi(x_1) + \Phi(x_2)\big).$$

Proposition 2.23. *T_mM is a vector space of dimension k.*

Proof. Pick a chart (U, φ) containing m and suppose $\varphi(m) = 0 \in \mathbb{R}^k$. Consider the map

$$\{\gamma \mid \gamma(0) = m\} \longrightarrow \mathbb{R}^k, \qquad \gamma \;\longmapsto \left.\frac{d}{dt}\right|_{t=0} \varphi \circ \gamma(t), \tag{2.24}$$

where γ is the smooth curve in M and, hence, $\varphi \circ \gamma(t)$ is the curve in $\mathbb{R}^k$.

Exercise 2.25. Show that this map is surjective.

If $\gamma_1 \sim \gamma_2$, then $\beta_1 := \varphi \circ \gamma_1 \sim \beta_2 := \varphi \circ \gamma_2$ so that $\dot{\beta}_1(0) = \dot{\beta}_2(0)$. Therefore, (2.24) induces a surjective map $\varphi_*\colon T_mM \to \mathbb{R}^k$, which is in fact bijective, since

$$\varphi_*[\gamma_1] = \varphi_*[\gamma_2] \quad \Longleftrightarrow \quad \beta_1 \sim \beta_2 \quad \Longleftrightarrow \quad \dot{\beta}_1(0) = \dot{\beta}_2(0).$$

Thus, we define the structure of a vector space on T_mM so that φ_* is a linear isomorphism.

Exercise 2.26. Show that the following holds:

(i) If $\gamma \in T_mM$ and $\lambda \in \mathbb{R}$, then $\lambda[\gamma] = [\gamma(\lambda \cdot)]$;

(ii) For two curves γ_1, γ_2 through m define

$$\gamma(t) := \varphi^{-1}\left(\beta_1(t) + \beta_2(t)\right),$$

where $\beta_j := \varphi \circ \gamma_j$. Show that γ is a smooth curve through m and

$$[\gamma_1] + [\gamma_2] = [\gamma].$$

We still need to show that the structure of the vector space on T_mM does not depend on the choice of the chart (U, φ). To this end, let $\big(\widehat{\mathrm{U}}, \widehat{\varphi}\,\big)$ be another chart such that $m \in \widehat{\mathrm{U}}$. Let

$$\widehat{\varphi}_*\colon T_mM \longrightarrow \mathbb{R}^k, \qquad [\gamma] \longmapsto \frac{d}{dt}\Big|_{t=0} \widehat{\varphi} \circ \gamma\,(t)$$

be the corresponding map. Denoting temporarily by $+_\varphi$ the addition obtained via φ_*, we obtain

$$[\gamma] = [\gamma_1] +_\varphi [\gamma_2] \qquad \Longleftrightarrow \qquad \dot\beta\,(0) = \dot\beta_1\,(0) + \dot\beta_2\,(0)\,,$$

where $\beta\,(t) = \varphi \circ \gamma\,(t)$ and $\beta_j\,(t) = \varphi \circ \gamma_j\,(t)$. Denote $\widehat{\beta}\,(t) = \widehat{\varphi} \circ \gamma\,(t)$ and $\widehat{\beta}_j\,(t) = \widehat{\varphi} \circ \gamma_j\,(t)$.

Then

$$\widehat{\beta} = \widehat{\varphi} \circ \gamma = \widehat{\varphi} \circ \varphi^{-1} \circ \varphi \circ \gamma = \theta \circ \beta \qquad \Longrightarrow \qquad \dot{\widehat{\beta}}\,(0) = D_p\theta\big(\dot\beta\,(0)\,\big).$$

Similarly, we have $\dot{\widehat{\beta}}_j\,(0) = D_p\theta\big(\dot\beta_j(0)\big)$.

Since $D_p\theta$ is a linear map, we have

$$\dot{\widehat{\beta}}\,(0) = D_p\theta\big(\dot\beta_1\,(0) + \dot\beta_2\,(0)\,\big) = D_p\theta\big(\dot\beta_1\,(0)\,\big) + D_p\theta\big(\dot\beta_2\,(0)\,\big) = \dot{\widehat{\beta}}_1\,(0) + \dot{\widehat{\beta}}_2\,(0)\,.$$

Hence, if $[\gamma] = [\gamma_1] +_\varphi [\gamma_2]$, then also $[\gamma] = [\gamma_1] +_{\widehat{\varphi}} [\gamma_2]$.

The fact that the multiplication with scalars is independent of the choice of a chart follows immediately from Exercise 2.26, (i). ☐

Notice that the origin in T_mM is represented by the constant curve $\gamma\,(t) = m$ (or any other curve equivalent to this one).

Remark 2.27. The proof of Proposition 2.23 implies the following. Let (U, φ) be a chart on M and $m \in \mathrm{U}$. Denote $\varphi\,(m) = p \in \mathbb{R}^k$ and define $\gamma_j\colon\ (-\varepsilon, \varepsilon) \to \mathrm{U}, \gamma_j\,(0) = m$ by

$$\varphi \circ \gamma_j\,(t) = p + (0, \dots, 0, t, 0, \dots, 0)\,.$$

where the non-trivial component is at the j^{th} place. Then

$$e_\varphi := \big(\,[\gamma_1]\,, \dots, [\gamma_k]\,\big) \tag{2.28}$$

is a basis of T_mM.

At his point for a surface $S \subset \mathbb{R}^3$ we have two definitions of the tangent space. The following proposition shows that these are equivalent.

Proposition 2.29. *If $S \subset \mathbb{R}^3$ is a smooth surface, then T_pS in the sense of Definition 2.21 is naturally isomorphic to the tangent plane of S.*

Proof. Denote temporarily the tangent plane of S at p in the sense of Definition 2.21 by E_p. Consider the map

$$T_pS \longrightarrow E_p, \qquad [\gamma] \longmapsto \dot\gamma\,(0)\,. \tag{2.30}$$

Exercise 2.31. Check that this map is well-defined, that is independent of the choice of the representative.

This map is linear. Indeed, if ψ is a parametrization at p such that $\psi(0) = p$ and

$$\gamma_1 := \psi \circ \beta_1, \qquad \gamma_2 := \psi \circ \beta_2$$

then $[\gamma_1] + [\gamma_2]$ is represented by the curve $t \mapsto \psi\big(\beta_1(t) + \beta_2(t)\big) = \gamma(t)$. Hence

$$\dot\gamma(0) = D_0\psi\big(\dot\beta_1(t) + \dot\beta_2(t)\big) = D_0\psi\big(\dot\beta_1(t)\big) + D_0\psi\big(\dot\beta_2(0)\big) = \dot\gamma_1(0) + \dot\gamma_2(0).$$

That is $[\gamma_1] + [\gamma_2] \in T_pS$ is mapped onto $\dot\gamma_1(0) + \dot\gamma_2(0)$.

Since $\lambda[\gamma] = [\gamma(\lambda\cdot)] \mapsto \frac{d}{dt}\Big|_{t=0}\gamma(\lambda t) = \lambda\dot\gamma(0)$, we see that (2.30) is linear. Since this map is clearly surjective and T_pS and E_p have equal dimension, (2.30) is an isomorphism. $\square$

Exercise 2.32 (The tangent space of a vector space)**.** Let V be a finite dimensional vector space. Show that for any $\mathrm{v} \in \mathrm{V}$ the map $\gamma \mapsto \dot\gamma(0)$, where γ is a smooth curve through v, induces a natural isomorphism $T_{\mathrm{v}}\mathrm{V} \to \mathrm{V}$. Thus, in the sequel we identify $T_{\mathrm{v}}V$ with V without further comments.

2.4 The differential of a smooth map

Let $f\colon M^k \to N^l$ be a smooth map.

Definition 2.33. For $m \in M$ the map

$$d_mf\colon T_mM \longrightarrow T_{f(m)}N, \qquad [\gamma] \longmapsto [f\circ\gamma]$$

is called the differential of f at m.

Proposition 2.34. *The differential is a linear map. If (U, φ) is a chart on M such that $m \in \mathrm{U}$ and (V, ψ) is a chart on N such that $f(m) \in \mathrm{V}$, then d_mf is represented by the Jacobi matrix of $F := \psi\circ f\circ\varphi^{-1}$ with respect to the bases e_φ and e_ψ, that is*

$$d_mf(e_\varphi) = e_\psi \cdot D_{\varphi(m)}F.$$

Proof. Assume for simplicity

$$\varphi(m) = 0 \in \mathbb{R}^k \qquad \text{and} \qquad \psi(f(m)) = 0 \in \mathbb{R}^l$$

For $\gamma\colon (-\varepsilon,\varepsilon) \to M$ such that $\gamma(0) = m$, denote $\beta := \varphi\circ\gamma\colon (-\varepsilon,\varepsilon) \to \mathbb{R}^k$.

We have

$$f\circ\gamma = f\circ\varphi^{-1}\circ\varphi\circ\gamma = f\circ\varphi^{-1}\circ\beta.$$

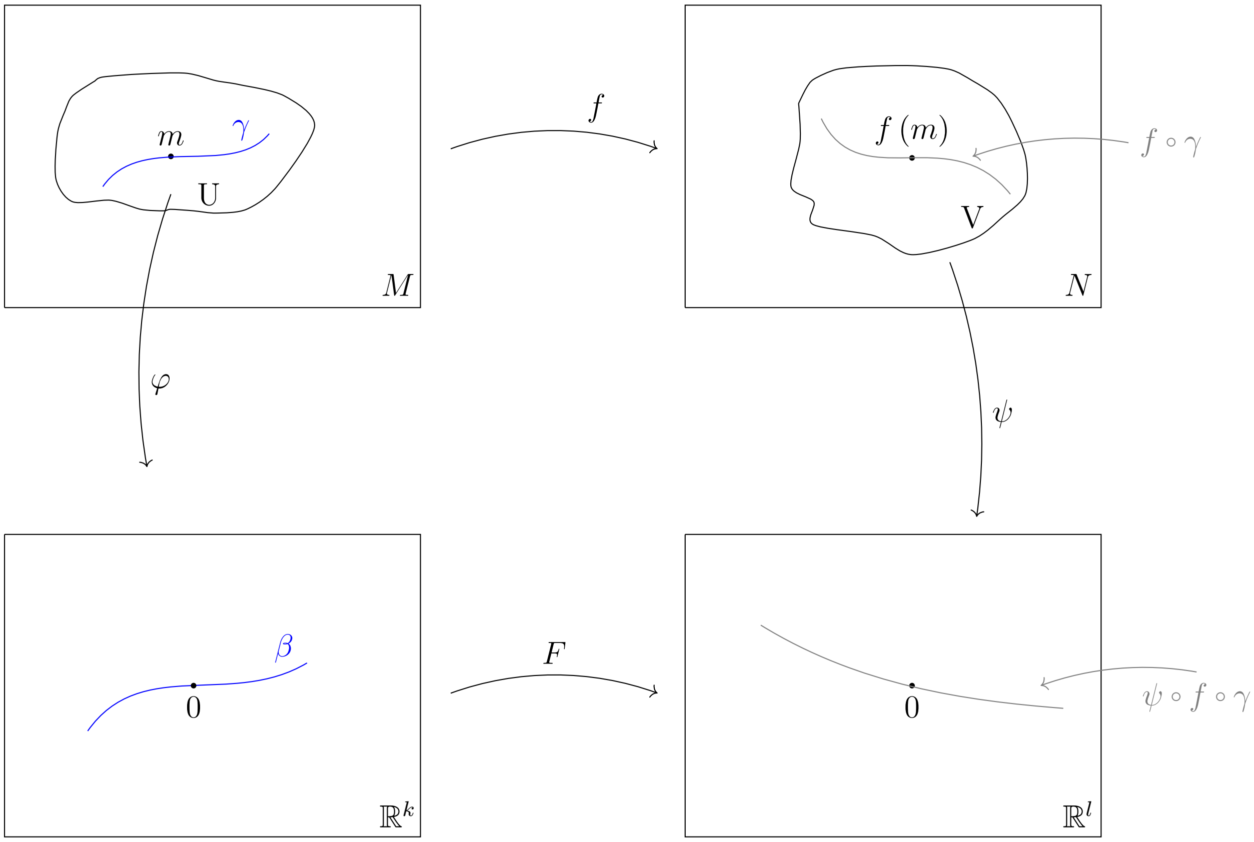
This yields

$$\psi_*[f\circ\gamma] = \frac{d}{dt}\Big|_{t=0}\Big(\underbrace{\psi\circ f\circ\varphi^{-1}}_{F}\circ\beta(t)\Big) = \frac{d}{dt}\Big|_{t=0}\big(F\circ\beta(t)\big) = D_0F\big(\dot\beta(0)\big).$$

By noticing the following $\dot\beta(0) = \frac{d}{dt}\Big|_{t=0}\varphi\circ\gamma(t) = \varphi_*[\gamma]$, we obtain

$$\psi_*d_mf([\gamma]) = D_0F\circ\varphi_*[\gamma] \qquad \forall\,\gamma \qquad \Longleftrightarrow \qquad d_mf \;\; = \psi_*^{-1}\circ D_0F\circ\varphi_*.$$

Since ψ_*^{-1}, D_0F, and φ_* are linear, we obtain that d_mf is linear too.



Furthermore, by the definition of e_φ, see (2.28), we have $\dot\beta_j(0) = (0, \dots, 0, 1, 0, \dots, 0) = e_j$. Hence,

$$\psi_* d_m f([\gamma_j]) = D_0 F(e_j) = \sum_{i=1}^{l} \frac{\partial F_i}{\partial x_j} \widehat{e}_i,$$

where $(\widehat{e}_1, \dots \widehat{e}_l)$ is a standard basis of $\mathbb{R}^l$. Hence,

$$d_m f[\gamma_j] = \psi_*^{-1}\left(\sum_{i=1}^{l} \frac{\partial F_i}{\partial x_j} \widehat{e}_i\right) = \sum_{i=1}^{l} \frac{\partial F_i}{\partial x_j} \underbrace{\psi_*^{-1}(\widehat{e}_i)}_{[\widehat{\gamma}_i]} \implies$$

$$\left(d_m f[\gamma_1], \dots, d_m f[\gamma_k]\right) = \left([\widehat{\gamma}_1], \dots, [\widehat{\gamma}_l]\right) \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \dots & \frac{\partial F_1}{\partial x_k} \\ \dots & \dots & \dots \\ \frac{\partial F_l}{\partial x_1} & \dots & \frac{\partial F_l}{\partial x_k} \end{pmatrix},$$

where $[\widehat{\gamma}_i]$ is the i^{th} element of e_ψ. This finishes the proof of this proposition. □

Exercise 2.35. Just as in Exercise 2.32, for a finite dimensional vector space V identify T_vV with V. Show that if $A\colon V \to W$ is a linear map, where W is another vector space, then $d_v A = A$.

Proposition 2.36. *For any smooth manifolds M, N, K and any $f \in C^\infty(M; N)$ and $g \in C^\infty(N; K)$ we have*

$$d_m(g \circ f) = d_{f(m)} g \circ d_m f.$$

Proof. For any $[\gamma] \in T_m M$ we have

$$d_m(g \circ f)[\gamma] = [g \circ f \circ \gamma] = [g \circ (f \circ \gamma)] = d_{f(m)} g([f \circ \gamma]) = d_{f(m)} g(d_m f[\gamma]),$$

where the third and fourth equalities follow from the definitions of $d_{f(m)}g$ and $d_m f$ respectively. $\square$

Corollary 2.37. *If $f\colon M \to N$ is a diffeomorphism, then $d_m f\colon T_m M \to T_{f(m)}N$ is an isomorphism. Conversely, if $d_m f$ is an isomorphism, then f is a local diffeomorphism at m.* $\square$

2.5 Submanifolds

Think of $\mathbb{R}^{k+l}$ as $\mathbb{R}^k \times \mathbb{R}^l$, where $k \geq 1, l \geq 1$. We have the maps

$$\begin{aligned} \iota_2\colon \mathbb{R}^l &\longrightarrow \mathbb{R}^{k+l}, & \iota_2(y) &= (0, y) \\ \pi_2\colon \mathbb{R}^{k+l} &\longrightarrow \mathbb{R}^l, & \pi_2(x, y) &= y \end{aligned}$$

where $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^l$.

Let $f\colon \mathbb{R}^{k+\ell} \to \mathbb{R}^\ell$ be a smooth map, which is defined on some neighbourhood U of the origin. For any point $p_0 = (x_0, y_0) \in U$ we have the following linear map

$$D_y f(p_0)\colon \mathbb{R}^\ell \xrightarrow{\iota_2} \mathbb{R}^{k+\ell} \xrightarrow{D_{p_0} f} \mathbb{R}^\ell. \tag{2.38}$$

For example, if $k = \ell = 1$, we have $D_y f(p_0) = \frac{\partial f}{\partial y}(p_0)$. For this reason, we call (2.38) *the partial derivative* of f with respect to y (at the point p_0).

To simplify the notations it is convenient to assume that p_0 is the origin and $f(0) = 0$, although this is immaterial.

Theorem 2.39. *If $D_y f(0)$ is an isomorphism, then there exists a smooth map $\theta\colon \mathbb{R}^{k+\ell} \to \mathbb{R}^{k+\ell}$, which is a local diffeomorpism at 0, such that $\theta(0) = 0$ and*

$$f \circ \theta = \pi_2$$

holds in a neighbourhood of the origin.

Proof. Define

$$g\colon \mathbb{R}^{k+\ell} \to \mathbb{R}^{k+\ell} \qquad \text{by} \qquad g(x, y) := \big(x, f(x, y)\big). \tag{2.40}$$

Then for the differential of g we have

$$Dg(0) = \begin{pmatrix} id_{\mathbb{R}^k} & 0 \\ D_x f(0) & D_y f(0) \end{pmatrix} \qquad \Longleftrightarrow \qquad Dg(0)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ D_x f(0)u + D_y f(0)v \end{pmatrix},$$

where $u \in \mathbb{R}^k$ and $v \in \mathbb{R}^\ell$.

If $(u\ v) \in \ker Dg(0)$, then $u = 0$ and $D_y f(0)v = 0$. However, $D_y f$ is an isomorphism by assumption of this theorem, so that $v = 0$. Therefore, $Dg(0)$ is injective and, hence, an isomorphism.

By the inverse map theorem, there is a local inverse $\theta\colon \mathbb{R}^{k+\ell} \to \mathbb{R}^{k+\ell}$ to g, that is in a neighbourhood of the origin we have

$$g \circ \theta = id_{\mathbb{R}^{k+\ell}} \qquad \Longrightarrow \qquad \pi_2 = \pi_2 \circ \mathrm{id}_{\mathbb{R}^{k+\ell}} = \pi_2 \circ g \circ \theta = f \circ \theta.$$

Thus, the theorem is proved. $\square$

Corollary 2.41 (The implicit function theorem)**.** *Suppose that the assumptions of Theorem 2.39 hold. Then there exists a neighbourhood V_1 of $0 \in \mathbb{R}^k$, a neighbourhood V_2 of $0 \in \mathbb{R}^\ell$, and a unique smooth map $h\colon V_1 \to V_2$ such that*

$$f(x,y) = 0 \quad \Longleftrightarrow \quad y = h(x) \tag{2.42}$$

whenever $(x,y) \in V_1 \times V_2$.

Furthermore, denoting $W := f^{-1}(0) \cap V_1 \times V_2$, the map

$$\psi := \pi_1\big|_W \colon W \to V_1, \qquad (x,y) \mapsto x$$

is a homemorphism, that is (W, ψ) is a chart on $f^{-1}(0)$ near the origin.

Proof. Let $\theta\colon U \to \theta(U)$ be the local diffeomorphism provided by Theorem 2.39. Pick any open subsets V_1 and V_2 as in the formulation of the theorem such that $V_1 \times V_2 \subset U$. For $x \in V_1$ define $h(x) := \pi_2 \circ \theta\,(x, 0)$. Furthermore, for $(x,y) \in V_1 \times V_2$, denote

$$(z,w) := \theta^{-1}(x,y) = g(x,y) = \big(x, f(x,y)\big).$$

Here we used the fact, that g, which is given by (2.40), is the inverse of θ. Then

$$\begin{aligned} f(x,y) = 0 \qquad &\Longrightarrow \qquad 0 = f\circ\theta\circ\theta^{-1}(x,y) = f\circ\theta\,(z,w) = w \\ &\Longrightarrow \qquad (z,0) = \big(x, f(x,y)\big). \end{aligned}$$

Hence, $z = x$ and $(x,y) = \theta(x,0)$, which yields in turn $y = h(x)$.

Furthermore, for any $x \in V_1$ we have

$$(x,0) = g\circ\theta\,(x,0) = g\big(\pi_1\circ\theta\,(x,0),\ \pi_2\circ\theta(x,0)\big)$$

From the definition of g we obtain $x = \pi_1 \circ \theta\,(x,0)$ and, hence, $0 = f\big(x, h(x)\big)$.

To show the uniqueness, notice that

$$\begin{aligned} f\big(x,\, h(x)\big) = 0 \qquad &\Longrightarrow \qquad g\big(x, h(x)\big) = \big(x, f\big(x, h(x)\big)\big) = (x,0) \\ f\big(x,\, \hat h(x)\big) = 0 \qquad &\Longrightarrow \qquad g\big(x, \hat h(x)\big) = (x,0). \end{aligned}$$

Since g is a local diffeomorphism, we obtain $h(x) = \hat h(x)$ provided x is sufficiently close to the origin.

Furthermore, notice that the map

$$V_1 \to W, \qquad x \mapsto \big(x, h(x)\big)$$

is a continuous inverse of ψ. Hence, ψ is a homeomorphism. □

The hypothesis of Corollary 2.41 implies that the differential of f at the origin is *surjective*. In fact, the surjectivity of the differential is decisive in Theorem 2.39 and Corollary 2.41, whereas the hypothesis that $D_y f(0)$ is an isomorphism can be achieved by a linear change of coordinates, see the proof of Theorem 2.47 below for some details.

Definition 2.43. Let N be a smooth manifold with an atlas $\mathcal{U}$. A chart (U, φ) is said to be smoothly compatible with $\mathcal{U}$ if $\mathcal{U} \cup \big\{(\mathrm{U}, \varphi)\big\}$ is again a smooth atlas.

The above definition simply means that both $\varphi \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ \varphi^{-1}$ are smooth for any chart $(\mathrm{U}_\alpha, \varphi_\alpha) \in \mathcal{U}$. Equivalently, this means that $\varphi\colon \mathrm{U} \to \varphi(\mathrm{U}) \subset \mathbb{R}^k$ is a diffeomorphism.

The proofs of Theorem 2.39 and Corollary 2.41 show that in the setting of these theorems there is a chart (U, φ) on $\mathbb{R}^{k+\ell}$ smoothly compatible with $\mathcal{U} = \big\{(\mathbb{R}^{k+l}, id)\big\}$ such that $\varphi\big(U \cap N\big) \subset \mathbb{R}^k \times \{0\}$. This motivates the following.

Definition 2.44. Let N be a manifold of dimension $k + \ell$. A subset $M \subset N$ is said to be *a submanifold* of dimension k (or k*-subamnifold*), if for each point $m \in M$ there exists a smoothly compatible chart (U, φ) on N centered at m such that

$$\varphi\big(U \cap M\big) = \varphi(U) \cap \big(\mathbb{R}^k \times \{0\}\big) \tag{2.45}$$

holds. Under these circumstances, the chart (U, φ) is said to be *adapted to* M.

Below we consider only charts smoothly compatible with a given atlas and therefore this will not be mentioned explicitly each time. Alternatively, given a smooth atlas $\mathcal{U}$ one can always replace $\mathcal{U}$ by the unique maximal atlas containing $\mathcal{U}$. Thus we can assume that $\mathcal{U}$ is maximal from the very beginning and in this case any chart smoothly compatible with $\mathcal{U}$ is contained in $\mathcal{U}$ so that one can simply talk about charts (from a maximal atlas).

Notice that if (U, φ) is an adapted chart, then $(M \cap U,\ \psi)$ is a chart on M, where

$$\psi\colon = \pi_1 \circ \varphi\big|_{U \cap M}\colon U \cap M \to \mathbb{R}^k.$$

Proposition 2.46. *A k-submanifold is a smooth k-manifold.*

Proof. By its very definition, a k-submanifold is equipped with a C^0-atlas $\mathcal{U}$, consisting of restrictions of all adapted charts.

I claim that this atlas is in fact smooth. Indeed, let (U_1, φ_1) and (U_2, φ_2) be two charts adapted to M. Denoting by $\imath_1\colon \mathbb{R}^k \to \mathbb{R}^{k+\ell}$ the inclusion $\imath_1(x) = (x, 0)$, we have

$$\psi_1 \circ \psi_2^{-1}(x) = \psi_1\big(\varphi^{-1}(x, 0)\big) = \pi_1 \circ \varphi_1 \circ \varphi_2^{-1} \circ \imath_1\ (x) = \pi_1 \circ \theta_{12} \circ \imath_1\ (x).$$

Thus, $\mathcal{U}$ is smooth. $\square$

We are now in the position to state one of the central theorems of this chapter.

Theorem 2.47. *Let M and N be smooth manifolds. If n is a regular value of a smooth map $f\colon M \to N$ and $\dim M \geq \dim N$, then $f^{-1}(n)$ is a submanifold of M of dimension $k := \dim M - \dim N$.*

Proof. Denote

$$\ell = \dim N \qquad \Longrightarrow \qquad \dim M = k + \ell.$$

Pick any $m \in f^{-1}(n)$ and any charts (U, φ) and (V, ψ) centered at m and n respectively. Let $F = \psi \circ f \circ \varphi^{-1}$ be the coordinate representation of f with respect to the charts (U, φ) and (V, ψ). Since φ and ψ are diffeomophisms, we obtain that the differential D_0F of F at the origin is surjective (in fact, D_pF is surjective at any point $p \in F^{-1}(0)$). In particular, $\dim\ker D_0F = k$.

Choose a basis $(\mathrm{v}_1, \dots, \mathrm{v}_{k+\ell})$ of $\mathbb{R}^k$ such that $(\mathrm{v}_1, \dots, \mathrm{v}_k)$ is a basis of $\ker D_0F$. Set

$$A\colon \mathbb{R}^{k+\ell} \to \mathbb{R}^{k+\ell}, \qquad z \mapsto \sum_{j=1}^{k+\ell} z_j \mathrm{v}_j.$$

Notice that by the definition of A and elementary facts from linear algebra, the following holds:

- A is an isomorphism;
- $A \circ \imath_1 \colon \mathbb{R}^k \to \ker D_0F$ is an isomorphism;
- $D_0F \circ A \circ \imath_2 \colon \mathbb{R}^\ell \to \mathbb{R}^\ell$ is an isomorphism.

Furthermore, consider the map $G := F \circ A \colon \mathbb{R}^{k+\ell} \to \mathbb{R}^\ell$. By Exercise 2.35, we have

$$D_0G = D_0F \circ A \qquad \Longrightarrow \qquad D_yG = D_0F \circ A \circ \imath_2.$$

Since the latter map is an isomorphism, by the proofs of Theorem 2.39 and Corollary 2.41 we obtain a chart (W, ξ) on $\mathbb{R}^{k+\ell}$ adapted to $G^{-1}(0)$, that is

$$\xi\big(W \cap G^{-1}(0)\big) = \xi(W) \cap \big(\mathbb{R}^k \times \{0\}\big).$$

Without loss of generality we can assume that W is contained in $A^{-1}\big(\varphi(U)\big)$.

Various charts involved in the proof are shown schematically on Figure 2.3.

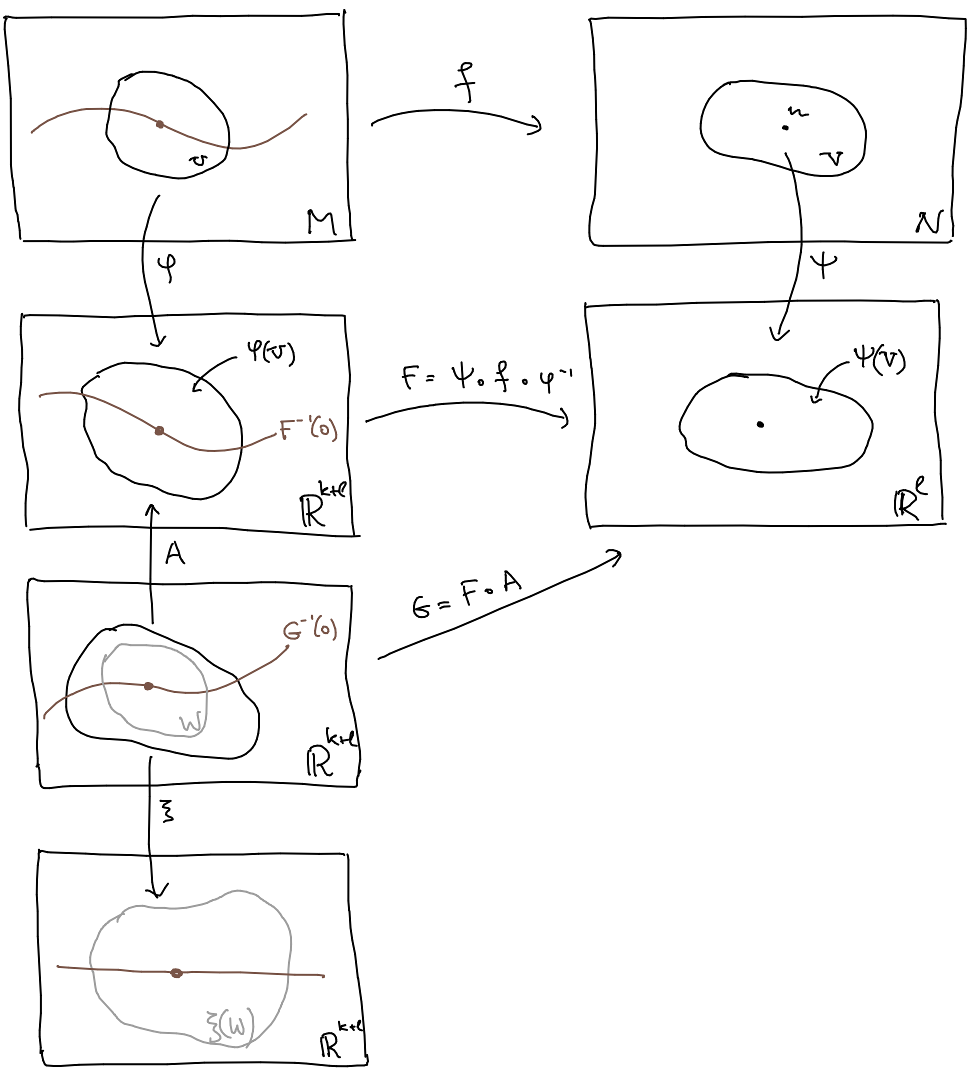


Figure 2.3: Scheme of the proof of Theorem 2.47.

Define a chart $(\hat W, \hat\xi\,)$ on $\mathbb{R}^{k+\ell}$ by

$$(\hat W, \hat\xi\,) = \big(\,A^{-1}(W),\ \xi\circ A^{-1}\,\big).$$

Since $z \in G^{-1}(0) \ \Leftrightarrow\ Az \in F^{-1}(0)$, we obtain

$$\hat\xi\big(\hat W \cap F^{-1}(0)\big) = \xi\big(W \cap G^{-1}(0)\big) = \xi(W)\cap\big(\mathbb{R}^k\times\{0\}\big).$$

Finally, setting

$$\varphi_1 := \hat\xi\circ\varphi \qquad\text{and}\qquad U_1 = \varphi_1^{-1}\big(\hat\xi(\hat W)\big) = \varphi^{-1}\big(\hat W\big)$$

we obtain

$$\varphi_1\big(U_1\cap f^{-1}(n)\big) = \hat\xi\big(\hat W\cap F^{-1}(0)\big) = \xi(W)\cap\big(\mathbb{R}^k\times\{0\}\big) = \varphi_1(U_1)\cap\big(\mathbb{R}^k\times\{0\}\big).$$

Thus, (U_1,φ_1) is a chart adapted to $f^{-1}(n)$ at m. $\square$

Notice the following: If $\dim M < \dim N$, then n is a regular value of smooth map $f\colon M\to N$ if and only if $n\notin \operatorname{Im} f$, see the paragraph following Definition 1.74. In this case $f^{-1}(n)=\varnothing$ is also (by definition) a smooth manifold. Thus, the condition $\dim M\geq\dim N$ can be dropped in the formulation of Theorem 2.47.

Proposition 2.48. *In the setting of Theorem 2.47, for any $m\in f^{-1}(n)$ we have*

$$T_m f^{-1}(n) = \ker d_m f.$$

Proof. Pick any curve γ in $f^{-1}(n)$ through m. Since γ lies in the level set of f, we have

$$f\circ\gamma\,(t) = n \qquad\text{for all } t\in(-\varepsilon,\varepsilon). \tag{2.49}$$

Since the constant curve $t\mapsto n$ represents the zero vector in T_nN, by the definition of the differential of f and (2.49) we obtain $d_mf\big([\gamma]\big)=0$. In other words any vector $[\gamma]$ tangent to $f^{-1}(n)$ lies in the kernel of d_mf. $\square$

Example 2.50.

(i) Consider the map $f\colon\mathbb{R}^{n+1}\to\mathbb{R}$, $f(x)=|x|^2$. Then 1 is a regular value of f. In particular, $S^n=f^{-1}(1)$ is a manifold of dimension n. Of course, the reader knows this fact by now very well.

(ii) Let $M_n(\mathbb{R})$ be the space of all $n\times n$ matrices with real entries. One can show that 1 is a regular value of the function $\det\colon M_n(\mathbb{R})\to\mathbb{R}$, $A\mapsto\det A$. Consequently,

$$\mathrm{SL}_n(\mathbb{R}) := \big\{A\in M_n(\mathbb{R})\mid \det A = 1\big\}$$

is a manifold of dimension $\dim M_n(\mathbb{R})-1 = n^2-1$.

Let us compute the tangent space to $\mathrm{SL}_n(\mathbb{R})$ at the point $\mathbb{1}$. To this end, it is convenient to identify $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$. Recalling that

$$\det A = \sum_\sigma \operatorname{sign}\sigma\, a_{1\,\sigma(1)}\dots a_{n\,\sigma(n)},$$

where σ runs through all permutations of the set $\{1, \dots, n\}$, for any $B \in M_n(\mathbb{R})$ we obtain

$$\det\big(\mathbb{1} + tB\big) = (1 + t\, b_{11})(1 + t\, b_{22}) \dots (1 + t\, b_{nn}) \\ + \sum_{\sigma \neq id} \operatorname{sign}\sigma\, \big(\delta_{1\,\sigma(1)} + tb_{1\,\sigma(1)}\big)\big(\delta_{2\,\sigma(2)} + tb_{2\,\sigma(2)}\big) \dots \big(\delta_{n\,\sigma(n)} + tb_{n\,\sigma(n)}\big).$$

Notice that for any $\sigma \neq id$, $\sigma(i) \neq i$ at least for two values of i. Hence, the last term in the above expression is $o(t)$. This yields

$$\det\big(\mathbb{1} + tB\big) = \big(1 + t\ \operatorname{tr} B + o(t)\big) + o(t).$$

Consequently, $\big(d_{\mathbb{1}} \det\big)\, B = \operatorname{tr} B$ and therefore

$$T_{\mathbb{1}} \mathrm{SL}_n(\mathbb{R}) = \big\{ B \in M_n(\mathbb{R}) \mid \operatorname{tr} B = 0 \big\}.$$

(iii) Let $Sym^n(\mathbb{R}) \subset M_n(\mathbb{R})$ denote the subspace of all symmetric matrices. One can show that the identity matrix $\mathbb{1} \in Sym^n(\mathbb{R})$ is a regular value of the map

$$f\colon M_n(\mathbb{R}) \to Sym^n(\mathbb{R}), \qquad f(A) = A \cdot A^t. \tag{2.51}$$

Consequently,

$$O(n) := \big\{ A \in M_n(\mathbb{R}) \mid A \cdot A^t = \mathbb{1} \big\}$$

is a manifold and

$$\dim O(n) = \dim M_n(\mathbb{R}) - \dim Sym^n(\mathbb{R}) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

Notice that if we would consider (2.51) as a map $M_n(\mathbb{R}) \to M_n(\mathbb{R})$, then $\mathbb{1}$ would not be a regular value.

Just like in the case of $\mathrm{SL}_n(\mathbb{R})$, let us compute the tangent space to $O(n)$ at the point $\mathbb{1}$. We have

$$f(\mathbb{1} + sB) = (\mathbb{1} + sB) \cdot (\mathbb{1} + sB)^t = \mathbb{1} + s\big(B + B^t\big) + o(s).$$

Hence, $d_{\mathbb{1}} f B = B + B^t$ and

$$T_{\mathbb{1}} O(n) = \big\{ B \in M_n(\mathbb{R}) \mid B^t = -B \big\}.$$

We finish this section by Sard's theorem, which, loosely speaking, says that for any smooth map almost any point is a regular value. More precisely, we say that a subset A of a smooth k-manifold M is of measure zero, if for any chart (U, φ) on M the set $\varphi\big(A \cap U\big) \subset \mathbb{R}^k$ is of measure zero.

Theorem 2.52 (Sard). *Let $f\colon M \to N$ be a smooth map between smooth manifolds. Then almost any point $n \in N$ is a regular value of f, that is the set of critical values for f is of measure zero.* □

A proof of Sard's theorem can be found for example in [BT03, 9.4] or [Mil65, §3].

2.6 Immersions and embeddings

Just like maps with surjective differentials can be conveniently described as projections after applying a diffeomorphisms, the maps with injective differentials admit an analogous description. We begin, however, with the following auxiliary result.

Theorem 2.53. *Let U be an open subset of $\mathbb{R}^k$ containing the origin and $f\colon U \to \mathbb{R}^k \times \mathbb{R}^\ell$ be a smooth map such that $f(0) = 0$ and*

$$D_0 f_1\colon \mathbb{R}^k \to \mathbb{R}^k, \qquad \textit{where} \;\; f_1 := \pi_1 \circ f,$$

is an isomorphism. Then there exists a neighbourhood $V \subset \mathbb{R}^{k+\ell}$ of the origin and a diffeomorphism $\theta\colon V \to \theta(V) \subset \mathbb{R}^{k+\ell}$ such that $\theta \circ f = \imath_1$ and $\theta\big(V \cap f(U)\big) = \theta(V) \cap \big(\mathbb{R}^k \times \{0\}\big)$.

Proof. The proof of this theorem is similar to the proof of Theorem 2.39.

Thus, consider the map

$$F\colon U \times \mathbb{R}^\ell \to \mathbb{R}^{k+\ell} = \mathbb{R}^k \times \mathbb{R}^\ell, \qquad F(x,y) := f(x) + (0,\, y) = \big(f_1(x),\; f_2(x) + y\big).$$

The differential of this map

$$D_0 F = \begin{pmatrix} D_0 f_1(0) & 0 \\ D_0 f_2(0) & \mathrm{id}_{\mathbb{R}^\ell} \end{pmatrix}$$

is an isomorphism. Hence, there exists a neighbourhood V of the origin and a diffeomorphism $\theta\colon V \to \theta(V)$ such that

$$\theta \circ F = \mathrm{id}_{\theta(V)}.$$

In particular, for any $(x, 0) \in \theta(V)$ the above equality yields:

$$\theta \circ F\,(x,0) = \theta \circ f(x) = \imath_1(x) \qquad \Longrightarrow \qquad \theta \circ f = \imath_1.$$

Hence, $\theta(V) \cap \big(\mathbb{R}^k \times \{0\}\big) \subset \theta\big(V \cap f(U)\big)$.

To show the converse inclusion, let $(x,y) \in \theta\big(V \cap f(U)\big)$. Hence, there exists some $(z,w) \in V \cap f(U)$ such that $(x,y) = \theta(z,w)$. In this case we must have $(z,w) = f(x)$ for some $x \in U$ and therefore

$$(x,y) = \theta(z,w) = \theta \circ f(x) = (x, 0).$$

Thus, $y = 0$ and $(x,0) \in V$, which yields $\theta\big(V \cap f(U)\big) \subset \theta(V) \cap \big(\mathbb{R}^k \times \{0\}\big)$. ☐

Definition 2.54. A smooth map $f\colon M^k \to N^\ell$ such that $d_m f$ is injective at each point $m \in M$ is called *an immersion*. An immersion, which is a diffeomorphism onto a k-submanifold of N, is called *an embedding*.

Clearly, an immersion $f\colon M \to N$ can exists only if $\dim M \le \dim N$. Notice also, that by Theorem 2.53 each immersion is locally injective, however an immersion does not need to be globally injective. Even if an immersion is injective, this may fail to be an embedding. This is shown schematically on Fugures 2.4 and 2.5 below. In particular, the image of an immersion does not need to be a submanifold.

Proposition 2.55. *An immersion which is a homeomorphism onto its image is an embedding.*

Proof. Denote $k := \dim M$ and $\ell := \dim N$. The proof consists of the following 3 steps.

Step 1. *For any $m \in M$ there exists a chart (V, ψ) on N centered at $n := f(m)$ with the following properties:*

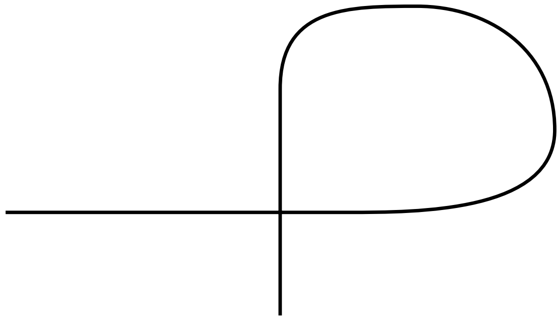


Figure 2.4: The image of a non-injective immersion $\mathbb{R} \to \mathbb{R}^2$.

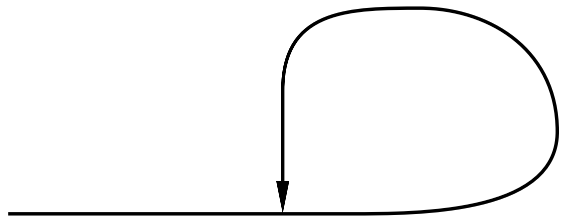


Figure 2.5: The image of an injective immersion $\mathbb{R} \to \mathbb{R}^2$, which is not an embedding.

- $d_n\psi_1\big|_{\operatorname{Im} d_m f}\colon \operatorname{Im} d_m f \to \mathbb{R}^k$ *is an isomorphism, where* $\psi_1 = \pi_1 \circ \psi$ *and* $\pi_1\colon \mathbb{R}^\ell = \mathbb{R}^k \oplus \mathbb{R}^{\ell-k} \to \mathbb{R}^k$ *is the projection.*
- *There exists a neighbourhood* U *of* m *such that* $f(U) = V \cap f(M)$.

Since f is a homeomorphism onto its image, $f\colon M \to f(M)$ is an open map. In particular, for any open $\hat U \subset M$ there exists an open subset $\hat V \subset N$ such that $f(\hat U) = \hat V \cap f(M)$. If $\hat U$ is a neighbourhood of m, we can choose a chart (V, ξ) centered at n such that $V \subset \hat V$.

Furthermore, since $d_n\xi\colon T_nN \to \mathbb{R}^\ell$ is an isomorphism and $\operatorname{Im} d_m f$ is a k-dimensional subspace of T_nN, we can find a linear isomorphism $A\colon \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that

$$A\Big(d_n\xi\big(\operatorname{Im} d_m f\big)\Big) = \mathbb{R}^k \times \{0\}.$$

Then $(V, \psi) = (V,\ A \circ \xi)$ is the required chart. Also, setting $U := f^{-1}(V)$ we obtain $f(U) = V \cap f(M)$.

Step 2. *$f(M)$ is a submanifold of N.*

Pick any $m \in M$ and a chart (U, φ) centered at m. Pick also a chart (V, ψ) as in the previous step. Denote also $W := \psi(V) \subset \mathbb{R}^\ell$.

Let $F = \psi \circ f \circ \varphi^{-1}$ be the coordinate representation of f. Denoting $F_1 := \pi_1 \circ F\colon \mathbb{R}^k \to \mathbb{R}^k$, we have

$$D_0F_1(0) = D_0\pi_1 \circ D_0F = D_0\pi_1 \circ d_n\psi \circ d_m f \circ d_0\varphi^{-1} = d_n\psi_1 \circ d_m f \circ d_0\varphi^{-1}$$

Since $d_0\varphi^{-1}$ is an isomorphism, by Step 1 we obtain that D_0F_1 is injective. Hence, D_0F_1 is an isomorphism. Hence, by Theorem 2.53 we can find a diffeomorphism[1] $\theta\colon W \to \theta(W) \subset \mathbb{R}^\ell$ such that

$$\theta \circ F = \imath_1 \qquad \Longleftrightarrow \qquad \big(\theta \circ \psi\big) \circ f \circ \varphi^{-1} = \imath_1$$

Denote $\hat\psi := \theta \circ \psi$. Then $(W, \hat\psi)$ is a chart on N adapted to $f(M)$.

Step 3. *f is a diffeomorphism between M and $f(M)$.*

[1]Without loss of generality we may assume that V was chosen so that θ is defined everywhere on W.

Let (U, φ) and $(W, \hat{\psi})$ be as in the preceding step. By the construction of $\hat{\psi}$, the coordinate representation of f is $\hat{\psi} \circ f \circ \varphi^{-1} = \imath_1 \colon \mathbb{R}^k \to \mathbb{R}^\ell$. Since the restriction of $\pi_1 \circ \psi$ to $f(M) \cap W$ is a chart on $f(M)$, the coordinate representation of f viewed as a map $f \colon M \to f(M)$ is given by

$$\pi_1 \circ \hat{\psi} \circ f \circ \varphi^{-1} = \pi_1 \circ \imath_1 = \mathrm{id}.$$

Hence, f is a local diffeomorphism. Since $f \colon M \to f(M)$ is bijective, this is a diffeomorphism. □

Corollary 2.56. *If M is compact, then any injective immersion $f \colon M \to N$ is an embedding.*

Proof. Pick a closed subset $A \subset M$. Since A is closed in M, A is compact and therefore $f(A)$ is compact in N. Since N is Hausdorff, $f(A)$ is closed. Hence, f is a closed map, i.e., the image of any closed subset is closed. This means that $f^{-1} \colon f(M) \to M$ is continuous, that is, $f \colon M \to f(M)$ is a homeomorphism. The statement of this corollary now follows immediately from Proposition 2.55. □

Theorem 2.47 combined with Sard's theorem allows us to construct many smooth manifolds, which are in fact submanifolds of Euclidean spaces. It turns out that any smooth manifold can be embedded into some Euclidean space, see, however, the discussion in the following section. Here we prove a version of this result in the case when the manifold under consideration is compact.

Theorem 2.57. *Any smooth compact manifold admits an embedding into some Euclidean space.*

Proof. For any $m \in M$ choose a chart (U_m, φ_m). Pick also open neigbourhoods $W_m \subset V_m$ and a bump function ρ_m such that the following holds:

- $\overline{V}_m \subset U_m$;
- $\rho_m\Big|_{\overline{W}_m} \equiv 1$ and $\rho_m < 1$ outside of $\overline{W}_m$;
- ρ_m vanishes outside of V_m.

Since M is compact, there is a finite subset $\{m_1, \dots, m_p\}$ of M such that $\{W_i\}$ cover all of M, where $W_i := W_{m_i}$. Consider each $\psi_i := \rho_i \cdot \varphi_i := \rho_{m_i} \cdot \varphi_{m_i}$ as a smooth map $M \to \mathbb{R}^k$ (extended by zero outside of V_m), where $k = \dim M$. Finally, define

$$f \colon M \to \mathbb{R}^{pk+p} \qquad \text{by} \qquad f(m) = \big(\psi_1(m), \dots, \psi_p(m), \rho_1(m), \dots, \rho_p(m)\big).$$

Clearly, f is smooth. I claim that this map is also injective. Indeed, pick any two distinct points m and $\hat{m}$. Without loss of generality, we can assume $m \in W_1$. If $\hat{m} \in \overline{W}_1$, then $\psi_1(m) = \varphi_1(m) \neq \varphi_1(\hat{m}) = \psi_1(\hat{m})$. If $\hat{m} \notin \overline{W}_1$, then $1 = \rho_1(m) \neq \rho_1(\hat{m})$, so that f is injective indeed.

Furthermore, assuming $m \in W_1$ again, $d_m\psi_1 \colon T_mM \to \mathbb{R}^k$ is an isomorphism, in particular, $d_m\psi_1$ is injective. Hence, $d_mf \colon T_mM \to \mathbb{R}^{kp+p}$ is injective at any $m \in M$. By Corollary 2.56, f is an embedding. □

2.7 The second countability axiom, the Whitney embedding theorem, and the existence of a partition of unity

Let $(X, \mathcal{T})$ be a topological space. Recall that a subset $\mathcal{B} \subset \mathcal{T}$ is called a basis of $\mathcal{T}$ if any point in X has a neighbourhood $U \in \mathcal{B}$. Equivalently, this means that any open subset in X can be represented as a union of subsets from $\mathcal{B}$.

Definition 2.58. X is said to satisfy the second axiom of countability (or, simply, X is second countable) if X admits a countable basis of its topology.

For example, the set $\{B_r(x) \mid r \in \mathbb{Q}_{>0},\ x \in \mathbb{Q}^n\}$ is a countable basis of the standard topology of $\mathbb{R}^n$. In particular, $\mathbb{R}^n$ is second countable.

Notice that any subspace of a second countable space is itself second countable.

Coming back to manifolds, notice that if a smooth manifold M can be embedded into some Euclidean space, M must be second countable. However, albeit any manifold according to Definition 2.5 is locally second countable, there may be no global countable base of its topology. Indeed, since any disjoint union of manifolds is again a manifold, the disjoint union of an uncountable family of manifolds is not second countable, and therefore does not admit an embedding into an Euclidean space. For this reason as well as some other ones, it is customary to restrict attention to second countable manifolds. With this in mind, from now on we replace Definition 2.5 by the following one.

Definition 2.59. A Hausdorff *second countable* topological space M is said to be a topological manifold of dimension $k \in \mathbb{N}_0$, if M is locally homeomorphic to $\mathbb{R}^k$. A smooth manifold is a topological manifold equipped with a smooth atlas (structure).

Of course, strictly speaking, at this point we should check that all constructions of manifolds we met before yield second countable manifolds if we start with second countable ones. I leave this as a (simple) exercise to the reader.

It turns out that adding the second countability to the definition of a manifold suffices to prove the following generalization of Theorem 2.57.

Theorem 2.60 (Whitney's embedding theorem)**.** *Any smooth manifold admits an embedding into some Euclidean space.* □

The proof of the above theorem is omitted here, but an interested reader may consult [War83] for a detailed discussion of these matters.

Another topic related to this one is the existence of a partition of unity (subordinate to a given covering). We have seen in the preceding chapter that partitions of unity are useful objects: Besides allowing one to define the notion of integral, this is an indispensable tool for various existence questions which were not discussed here because of lack of time. I just mention two examples here: Using the existence of partitions of unity one can show, and rather trivially, that any abstract manifold admits a Riemannian metric and that any smooth function on a submanifold can be obtained as a restriction of a smooth function on an ambient manifold.

Observe that Whitney's embedding theorem implies the existence of partitions of unity. Indeed, we have seen that $\mathbb{R}^n$ admits a partition of unity subordinate to any open covering. If M is embedded into $\mathbb{R}^n$, one can simply restrict a given partition of unity to M to obtain the existence on M.

Exercise 2.61. Check that the proof of Theorem 1.101 goes through for any compact manifold thus proving directly that compact manifolds admit partitions of unity subordinate to any given open covering.

Whitney's embedding theorem shows that any (abstract) manifold M can be thought of as a submanifold of an Euclidean space. In other words, we could have defined manifolds as subspaces of Euclidean spaces admitting charts[2] at each point. Some authors do take this point

[2]More precisely, admitting adapted charts.

of view arguing that this yields the same pool of examples. While it is true of course that this yields the same pool of examples, manifolds often do not arise as subsets of Euclidean spaces. For example, the real projective space is not obviously contained in any Euclidean space (and it is even not so obvious how to construct an embedding). Even if one decides to work with submanifolds of Euclidean spaces only, one finds pretty soon that certain useful constructions, for example taking quotients by group actions, are incompatible with this setting. More importantly, it is useful to distinguish "inner" properties of manifolds from those of an embedding. All these reasons led to the necessity to separate abstract manifolds from their particular realizations as submanifolds.

2.8 The tangent bundle

It is convenient to recall certain constructions from linear algebra first. Thus, let V be a vector space of dimension k. Any basis $\mathrm{v} = (\mathrm{v}_1, \dots, \mathrm{v}_k)$ of V yields an isomorphism

$$\mathbb{R}^k \longrightarrow \mathrm{V}, \qquad y \longmapsto \sum_{j=1}^{k} y_j \mathrm{v}_j = \mathrm{v} \cdot y$$

Conversely, if $\varphi\colon \mathbb{R}^k \longrightarrow \mathrm{V}$ is a linear isomorphism, then the image of the standard basis of $\mathbb{R}^k$ is a basis of V. This yields a bijective correspondence between the set of all bases of V and the set of all isomorphisms $\mathbb{R}^k \to \mathrm{V}$.

If $\mathrm{w} = (\mathrm{w}_1, \dots, \mathrm{w}_k)$ is another basis of V, we obtain the change-of-basis matrix B as follows. Writing

$$\mathrm{w}_j = \sum_{i=1}^{k} b_{ij} \mathrm{v}_i \tag{2.62}$$

we set $B = (b_{ij})$. Then (2.62) is equivalent to

$$\mathrm{w} = \mathrm{v} \cdot B$$

where $\cdot$ represents matrix multiplication.

With these preliminaries at hand, suppose that M is a manifold of dimension k. Pick a point $m \in M$ and a chart (U, φ) such that $m \in \mathrm{U}$. Denote $p := \varphi(m) \in \mathbb{R}^k$. We obtain a basis of $T_m M$ as follows:

$$\mathrm{v}_\varphi = \mathrm{v} := \big([\gamma_1], \dots, [\gamma_k] \big), \tag{2.63}$$

where $\gamma_j(t) = \varphi^{-1}(p + te_j)$ and $e = (e_1, \dots, e_k)$ is the standard basis of $\mathbb{R}^k$, see Remark 2.27.

Remark 2.64. As we have seen in the preceding chapter, for a surface $S \subset \mathbb{R}^3$ a choice of a parametrization $\psi = \psi(u, v)$ yields a basis $\big(\partial_u \psi,\ \partial_v \psi\big)$ of $T_{\psi(u,v)} S$. For abstract manifolds (2.63) plays a rôle similar to the one $\big(\partial_u \psi,\ \partial_v \psi\big)$ plays for surfaces.

If $\big(\widehat{\mathrm{U}}, \widehat{\varphi}\big)$ is another chart such that $m \in \widehat{\mathrm{U}}$, we obtain another basis

$$\mathrm{v}_{\widehat{\varphi}} = \widehat{\mathrm{v}} := \big([\widehat{\gamma}_1], \dots, [\widehat{\gamma}_k] \big),$$

where $\widehat{\gamma}_j(t) = \widehat{\varphi}^{-1}(\widehat{p} + te_j)$ and $\widehat{p} = \widehat{\varphi}(m)$.

Proposition 2.65. *Let $\theta := \widehat{\varphi} \circ \varphi^{-1}\colon \mathbb{R}^k \longrightarrow \mathbb{R}^k$ be the coordinate transformation map. Then the change-of-basis matrix between v and $\widehat{\mathrm{v}}$ is $D_p\theta$: $\mathrm{v} = \widehat{\mathrm{v}} \cdot D_p\theta$.*

Proof. Without loss of generality we can assume $p = 0 = \widehat{p}$. We have

$$\widehat{\varphi} \circ \gamma_j(t) = \underbrace{\widehat{\varphi} \circ \varphi^{-1}}_{\theta} \circ \underbrace{\varphi \circ \gamma_j(t)}_{te_j} = \theta(te_j).$$

Hence,

$$d_m\widehat{\varphi}[\gamma_j] = \frac{d}{dt}\Big|_{t=0} \theta(te_j) = \sum_{i=1}^{k} \frac{\partial \theta_i}{\partial x_j} e_i, \tag{2.66}$$

where the partial derivatives are evaluated at the origin (suppressed in the notations).

Notice, however,

$$\widehat{\varphi} \circ \widehat{\gamma}_i(t) = te_i \qquad \Longrightarrow \qquad d_m\widehat{\varphi}[\widehat{\gamma}_i] = e_i.$$

Hence, by (2.66) we obtain

$$d_m\widehat{\varphi}[\gamma_j] = \sum_{i=1}^{k} \frac{\partial \theta_i}{\partial x_j} d_m\widehat{\varphi}([\widehat{\gamma}_i]) = d_m\widehat{\varphi}\left(\sum_{i=1}^{k} \frac{\partial \theta_i}{\partial x_j}[\widehat{\gamma}_i]\right).$$

where the second equality holds by the linearity of $d_m\widehat{\varphi}$.

Since $\widehat{\varphi}\colon \widehat{\mathrm{U}} \to \widehat{\varphi}(\widehat{\mathrm{U}}) \subset \mathbb{R}^k$ is a diffeomorphism, $d_m\widehat{\varphi}$ is an isomorphism. Hence,

$$[\gamma_j] = \sum_{i=1}^{k} \frac{\partial \theta_i}{\partial x_j}[\widehat{\gamma}_i],$$

which finishes the proof. $\square$

Consider the set

$$TM = \bigsqcup_{m \in M} T_mM,$$

where the symbol $\sqcup$ denotes the disjoint union.

This comes equipped with the map

$$\pi\colon TM \longrightarrow M, \qquad \pi(\mathrm{v}) = m \Longleftrightarrow \mathrm{v} \in T_mM.$$

Example 2.67. Just as in Exercise 2.32 for a vector space V with the help of the canonical isomorphism $T_{\mathrm{v}}\mathrm{V} \cong \mathrm{V}$ we obtain

$$T\mathrm{V} = \bigsqcup_{\mathrm{v} \in \mathrm{V}} \{\mathrm{v}\} \times \mathrm{V} = \mathrm{V} \times \mathrm{V}$$

and $\pi(\mathrm{v}, \mathrm{w}) = \mathrm{v}$ is the projection onto the first component.

Furthermore, for any chart (U, φ) on M we have a basis $\mathrm{v}_{\varphi(m)}$ of T_mM for each $m \in \mathrm{U}$. Therefore, we obtain the bijection

$$\mathrm{U} \times \mathbb{R}^k \longrightarrow \pi^{-1}(\mathrm{U}) = \bigsqcup_{m \in \mathrm{U}} T_mM, \qquad (m, y) \longmapsto \mathrm{v}_{\varphi}(m) \cdot y = \sum_{j=1}^{k} y_j\left[\gamma_j^m\right],$$

where $\gamma_j^m(t) = \varphi^{-1}(\varphi(m) + te_j)$. Combining this with $\varphi\colon \mathrm{U} \to \varphi(\mathrm{U})$, which is also a bijection, we obtain a bijective map

$$\tau = \tau_{\varphi}\colon \varphi(\mathrm{U}) \times \mathbb{R}^k \longrightarrow \pi^{-1}(\mathrm{U}), \qquad (x, y) \longmapsto \mathrm{v}_{\varphi}(m) \cdot y = \sum y_j\left[\gamma_j^m\right],$$

where $m = \varphi^{-1}(x)$.

Theorem 2.68. *Let $\mathcal{U} = \left\{ (\mathrm{U}_\alpha, \varphi_\alpha) \mid \alpha \in A \right\}$ be a smooth atlas on M. There exists a unique Hausdorff second countable topology on TM such that*

$$\mathcal{V} = \left\{ \left(\pi^{-1}\left(\mathrm{U}_\alpha\right), \tau_\alpha^{-1} \right) \mid \alpha \in A \right\}$$

is a C^0-atlas on TM, where $\tau_\alpha = \tau_{\varphi_\alpha}$. In fact, $\mathcal{V}$ is a smooth atlas so that TM is a smooth manifold of dimension $2k$. Moreover, π is a smooth map with surjective differential at each point.

Proof. The proof consists of the following steps.

Step 1. *For the coordinate transformation $\Theta_{\alpha\beta} = \tau_\alpha^{-1} \circ \tau_\beta$ on TM we have*

$$\Theta_{\alpha\beta}\left(x, y\right) = \left(\theta_{\alpha\beta}\left(x\right), D_x\theta_{\alpha\beta} \cdot y\right).$$

In particular, $\Theta_{\alpha\beta}$ is smooth.

Denote $\tau_\beta\left(x, y\right) = \mathrm{v}$ that is $\varphi_\beta\left(\pi\left(\mathrm{v}\right)\right) = x$ and $\mathrm{v} = \mathrm{v}_\beta\left(\varphi_\beta^{-1}\left(x\right)\right) \cdot y$. Denoting $m = \varphi_\beta^{-1}\left(x\right)$ and recalling Proposition 2.65, we obtain

$$\mathrm{v}_\beta\left(m\right) = \mathrm{v}_\alpha\left(m\right) D_x\theta_{\alpha\beta}.$$

Denoting also $\tau_\alpha^{-1}\left(\mathrm{v}\right) = \left(s, t\right) \in \mathbb{R}^k \times \mathbb{R}^k$. Since $\mathrm{v} = \mathrm{v}_\beta\left(m\right) \cdot y = \mathrm{v}_\alpha \cdot D\theta_{\alpha\beta} \cdot y$, we obtain

$$s = \varphi_\alpha^{-1}\left(\pi\left(\mathrm{v}\right)\right) = \varphi_\alpha\left(\varphi_\beta^{-1}\left(x\right)\right) = \theta_{\alpha\beta}\left(x\right) \qquad \text{and} \qquad t = D\theta_{\alpha\beta} \cdot y.$$

Step 2. *There is a unique Hausdorff second countable topology on TM such that each τ_α is a homeomorphism onto its image.*

Declare a set $\mathrm{V} \subset TM$ open if and only if $\tau_\alpha^{-1}\left(\mathrm{V}\right)$ is open in $\mathbb{R}^{2k}$ for any $\alpha \in A$. We have

(i) $\varnothing$ is open and $\tau_\alpha^{-1}\left(TM\right) = \varphi_\alpha\left(\mathrm{U}_\alpha\right) \times \mathbb{R}^k$ is open.

(ii) $\mathrm{V}_1, \mathrm{V}_2$ are open $\Rightarrow \tau_\alpha^{-1}\left(\mathrm{V}_1 \cap \mathrm{V}_2\right) = \tau_\alpha^{-1}\left(\mathrm{V}_1\right) \cap \tau_\alpha^{-1}\left(\mathrm{V}_2\right)$ is open $\Rightarrow \mathrm{V}_1 \cap \mathrm{V}_2$ is open

(iii) Each $\mathrm{V}_\beta, \beta \in B$, is open $\Rightarrow \tau_\alpha^{-1}\left(\cup_{\beta \in B} \mathrm{V}_\beta\right) = \cup_{\beta \in B} \tau_\alpha^{-1}\left(\mathrm{V}_\beta\right)$ is open $\Rightarrow \cup_{\beta \in B} \mathrm{V}_\beta$ is open.

Hence, we obtain a topology on TM such that each $\left(\pi^{-1}\left(\mathrm{U}_\alpha\right), \tau_\alpha^{-1}\right)$ is a chart on TM and, moreover, π is a continuous map.

This topology is Hausdorff. Indeed, pick $\mathrm{v}_1, \mathrm{v}_2 \in TM, \mathrm{v}_1 \neq \mathrm{v}_2$ and consider the following cases:

(a) If $\pi\left(\mathrm{v}_1\right) \neq \pi\left(\mathrm{v}_2\right)$, choose open subsets $\mathrm{U}_1, \mathrm{U}_2 \subset M$ such that U_1 and U_2 separate $\pi\left(\mathrm{v}_1\right)$ and $\pi\left(\mathrm{v}_2\right)$. Then $\pi^{-1}\left(\mathrm{U}_1\right)$ and $\pi^{-1}\left(\mathrm{U}_2\right)$ separate $\pi\left(\mathrm{v}_1\right)$ and $\pi\left(\mathrm{v}_2\right)$.

(b) If $\pi\left(\mathrm{v}_1\right) = \pi\left(\mathrm{v}_2\right) =: m$. Pick any chart $\left(\mathrm{U}, \varphi\right)$ such that $m \in \mathrm{U}$. Then $\tau\left(\mathrm{U} \times \mathrm{V}_1\right)$ and $\tau\left(\mathrm{U} \times \mathrm{V}_2\right)$ separate v_1 and v_2 if $\mathrm{V}_1, \mathrm{V}_2 \subset \mathbb{R}^k$ separate $\pi_2\left(\tau^{-1}\left(\mathrm{v}_1\right)\right)$ and $\pi_2\left(\tau^{-1}\left(\mathrm{v}_2\right)\right)$.

Furthermore, the constructed topology is second countable for the following reason: Let U_i be a countable basis of the topology of M and V_j be a countable basis for $\mathbb{R}^k$. Without loss of generality we can assume that each U_i is contained in some chart U_{α_i}. Then the collection of all sets of the form

$$\tau_{\alpha_i}\left(\varphi_{\alpha_i}(\mathrm{U}_i) \times \mathrm{V}_j\right)$$

is a countable basis for the topology of TM.

Step 3. *We finish the proof of this theorem.*

Pick a chart $(\mathrm{U}_\alpha, \varphi_\alpha)$ on M, hence also a chart $\left(\pi^{-1}(\mathrm{U}_\alpha), \tau_\alpha^{-1}\right)$ on TM. The coordinate representation of π with respect to these charts is

$$\varphi_\alpha \circ \pi \circ \tau_\alpha (x, y) = \varphi_\alpha \left(\varphi_\alpha^{-1}(x)\right) = x \qquad \Longrightarrow \qquad \varphi_\alpha \circ \pi \circ \tau_\alpha = \pi_1.$$

Hence, π is smooth and $d_{\mathrm{v}}\pi$ is surjective. $\square$

Let $M \subset \mathbb{R}^n$ be a submanifold. To simplify the exposition somewhat, assume that $M = f^{-1}(0)$, where $f \in C^\infty(\mathbb{R}^n)$ and 0 is a regular value of f (and, hence, M is a hypersurface).

Let v be any tangent vector at $m \in M$ represented by a curve γ through m. Thinking of γ as a curve in $\mathbb{R}^n$, we obtain a natural well-defined map $p\colon TM \to \mathbb{R}^n$ given by $p(\mathrm{v}) = \dot\gamma(0)$. Combining this with the projection π, we obtain the map

$$i\colon TM \longrightarrow \mathbb{R}^n \times \mathbb{R}^n, \qquad i(\mathrm{v}) = \big(\pi(\mathrm{v}),\ p(\mathrm{v})\big).$$

Define $\Phi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^2$ by $\Phi(x, y) = \big(f(x),\ \langle \nabla f(x),\ y\rangle\big)$. The reader should check that 0 is a regular value of Φ. Moreover, $\Phi^{-1}(0) = i(TM)$ and i is an embedding. In particular, the tangent bundle of a hypersurface is an embedded submanifold.

For example, for $M = S^k \subset \mathbb{R}^{k+1}$, we have

$$TS^k = \big\{(x, y) \in S^k \times \mathbb{R}^{k+1} \mid \langle x, y\rangle = 0\big\} \subset \mathbb{R}^{2k+2}.$$

In particular, for $k = 1$ we obtain that TS^1 is a 2-submanifold of $\mathbb{R}^4$.

In fact, we can realize TS^1 as a submanifold of $\mathbb{R}^3$ in the following sense. Consider the map

$$f\colon S^1 \times \mathbb{R} \to \mathbb{R}^4, \qquad f(x_0, x_1; t) = (x_0, x_1, tx_1, -tx_0).$$

One can check that f is a diffeomorphism between $S^1 \times \mathbb{R} \subset \mathbb{R}^3$ and TS^1 so that we can in fact identify TS^1 with an infinite cylinder.

It is not too hard to generalize the above arguments to show that if $M \subset \mathbb{R}^n$ is any embedded submanifold, then $i\colon TM \to \mathbb{R}^{2n}$ is an embedding. I leave it to the reader to work out the details of this statement.

2.9 Vector fields and their integral curves

Definition 2.69. A smooth map $v\colon M \to TM$ such that

$$\pi \circ v = id_M \qquad \Longleftrightarrow \qquad v(m) \in T_m M$$

is called a (smooth) *vector field* on M.

For example, the map

$$v\colon S^1 \to \mathbb{R}^2, \qquad v(x) = \big(x,\ (-x_1, x_0)\big)$$

is a (smooth) vector field on S^1. Since the first component of v must be x by the very definition of a vector field, usually one omits the first component and writes simply

$$v(x) = (-x_1, x_0). \tag{2.70}$$

Denote

$$\mathfrak{X}(M) := \big\{v\colon M \to TM \text{ is a vector field }\big\}.$$

Clearly, $\mathfrak{X}(M)$ is a real vector space with respect to the following operations:

- $\big(v_1+v_2\big)(m) := v_1(m)+v_2(m)$, where $v_1, v_2 \in \mathfrak{X}(M)$;
- $\big(\lambda v\big)(m) = \lambda v(m)$, where $v \in \mathfrak{X}(M)$ and $\lambda \in \mathbb{R}$.

In fact, any vector field can be multiplied by any smooth function:

$$\big(f\cdot v\big)(m) = f(m)v(m), \qquad \text{where } v \in \mathfrak{X}(M) \text{ and } f \in C^\infty(M).$$

We summarize this in the following.

Proposition 2.71. *The set $\mathfrak{X}(M)$ of all vector fields on M has the structure of a module over $C^\infty(M)$ with respect to the pointwise addition and multiplication.* □

Example 2.72. Consider $M = \mathbb{R}^k$. We have seen that $T\mathbb{R}^k \cong \mathbb{R}^k \times \mathbb{R}^k$ and that the natural projection equals π_1. Hence, a vector field is a map of the form

$$v(x) = \big(x, y(x)\big),$$

where $y \in C^\infty(\mathbb{R}^k;\mathbb{R}^k)$. Hence, we can identify $\mathfrak{X}(\mathbb{R}^k)$ with $C^\infty(\mathbb{R}^k;\mathbb{R}^k)$ via the map

$$v = \big(id_{\mathbb{R}^k},\, y\big) \mapsto y.$$

More formally, this map is an isomorphism of $C^\infty(M)$-modules.

Generalizing the above example slightly, pick a chart (U,φ) on a manifold M. Since

$$\mathrm{v}_\varphi(m) := \big([\gamma_1^m],\dots,[\gamma_k^m]\big), \qquad \text{where} \quad \gamma_j^m(t) := \varphi^{-1}\big(\varphi(m)+te_j\big),$$

is a basis of T_mM, we can find the coordinates $\big(y_1(m),\dots,y_k(m)\big)$ of $v(m)$ with respect to this basis. In other words, $y\colon U \to \mathbb{R}^k$ is a map such that

$$v(m) = \mathrm{v}_\varphi(m)\cdot y(m)$$

holds at any point $m \in U$. Notice that the map y is well defined even if v is not necessarily smooth. This map is called *the coordinate (or local) representation of* v with respect to the chart (U,φ).

Proposition 2.73. *The map $v\colon M \to TM$ satisfying $\pi\circ v = id_M$ is a smooth vector field if an only if for each chart (U,φ) as above the coordinate representation y of v is smooth.*

Proof. Recall that for any chart (U,φ) on M as above we constructed a chart $\big(\pi^{-1}(U),\ \tau_\varphi^{-1}\big)$ on TM. Just by the definitions of τ_φ and y, for the coordinate representation of v with respect to these charts we have

$$\tau_\varphi^{-1}\circ v\circ\varphi^{-1}(x) = \big(x,\ y\circ\varphi^{-1}(x)\big).$$

Hence, v is smooth if and only if y is smooth. □

Thus, locally over each chart U vector fields can be identified with smooth vector-valued maps just as in Example 2.72. It turns out, however, that in general no such identification can exist.

Let $\gamma\colon (a,b) \to M$ be a smooth curve. At any point $t \in (a,b)$ we define *the tangent vector* $\dot\gamma(t) \in T_{\gamma(t)}M$ to γ by

$$\dot\gamma(t) := \big[s \mapsto \gamma_t(s)\big] \qquad \text{where} \quad \gamma_t(s) := \gamma(t+s).$$

Definition 2.74 (Integral curves)**.** A (smooth) curve γ is called *an integral curve* of a vector field v if

$$\dot\gamma(t) = v\big(\gamma(t)\big)$$

holds for any $t \in (a, b)$.

Example 2.75. Consider the curve $\gamma\colon \mathbb{R} \to S^1$, $\gamma(t) = (\cos t,\ \sin t)$. We have $\dot\gamma(t) = (-\sin t,\ \cos t)$. Furthermore, if v is given by (2.70), then

$$v \circ \gamma\,(t) = (-\sin t,\ \cos t).$$

Hence, γ is an integral curve of (2.70).

Let us consider integral curves on $\mathbb{R}^k$ in some detail. Thus, represent a vector field $v \in \mathfrak{X}(\mathbb{R}^k)$ by a smooth map $y\colon \mathbb{R}^k \to \mathbb{R}^k$ just as in Example 2.72 above. A map $\gamma\colon (a, b) \to \mathbb{R}^k$ is an integral curve of v if and only if

$$\dot\gamma(t) = y\big(\gamma(t)\big) \qquad \iff \qquad \begin{cases} \dot\gamma_1(t) = y_1\big(\gamma_1(t), \dots, \gamma_k(t)\big),\\ \qquad \cdots\cdots\cdots \\ \dot\gamma_k(t) = y_k\big(\gamma_1(t), \dots, \gamma_k(t)\big), \end{cases} \tag{2.76}$$

holds for any $t \in (a, b)$. In other words, an integral curve of a vector field is a solution of a system of ordinary differential equations (ODEs). Notice that the map y does not depend on t, that is (2.76) is *an autonomous* system of ODEs.

Conversely, any system of ODEs as above, is uniquely specified by a map $y \in C^\infty(\mathbb{R}^k; \mathbb{R}^k)$. In view of Example 2.72, y corresponds to a vector field v, whose integral curves are solutions of the initial system of ODEs. Thus, at least for Euclidean spaces, integral curves of vector fields and solutions of autonomous systems of ODEs are synonymous.

Exercise 2.77. Show that if γ is a C^1-curve satisfying (2.76), then γ is smooth.

Notice that for autonomous systems we have the following property: If γ is a solution of (2.76) such that $\gamma(t_0) = m_0$, then for any $c \in (a, b)$

$$\gamma_c(t) := \gamma(t + c), \qquad t \in (a - c, b - c)$$

is also a solution. In other words, the integral curve γ_1 of v such that $\gamma_1(t_1) = m_0$ satisfies

$$\gamma_1(t) = \gamma\big(t + t_0 - t_1\big),$$

that is γ_1 differs from γ just by a shift of time. For this reason, one often chooses $t_0 = 0$ as the initial time for integral curves of vector fields.

By the main theorem of ODEs [Hal80, Sec.I.3], we obtain the following existence and uniqueness result.

Theorem 2.78. *Let v be a smooth vector field on an open subset $\Omega \subset \mathbb{R}^k$. For any point $m_0 \in \Omega$ there exists a neighbourhood $V \subset \Omega$ of m_0 and a number $\varepsilon > 0$ with the following property: For any $m \in V$ there exists an integral curve*

$$\gamma = \gamma_m\colon (-\varepsilon, \varepsilon) \to \Omega \qquad \text{such that} \qquad \gamma(0) = m.$$

This integral curve is unique in the following sense: If $\beta\colon (-\delta, \delta) \to M$ is any other integral curve such that $\beta(0) = m$, then β and γ_m coincide on $(-\varepsilon, \varepsilon) \cap (-\delta, \delta)$. Moreover, the map

$$\Phi\colon (-\varepsilon, \varepsilon) \times V \to \mathbb{R}^k, \qquad \Phi(t, m) := \gamma_m(t) \tag{2.79}$$

is smooth. □

Definition 2.80. An integral curve $\gamma\colon (a,b) \to M$ of a vector field v is called *maximal*, if the following property holds: For any other integral curve $\beta\colon (c,d) \to M$ of v such that for some $t_0 \in (a,b) \cap (c,d)$ we have $\gamma(t_0) = \beta(t_0)$, then:

(i) $(c,d) \subset (a,b)$;

(ii) $\beta = \gamma\big|_{(c,d)}$.

It is a well-known fact from the theory of ODEs, that for any $m_0 \in \mathbb{R}^k$ there is a unique maximal solution of (2.76) through m_0. A straightforward corollary is, that for any vector field v on any manifold M there is a unique maximal integral curve γ of v through a given point.

Corollary 2.81. *If M is compact, then a maximal integral curve of any vector field is defined on all of $\mathbb{R}$.*

Proof. For each point $m \in M$ pick a chart (U,φ) containing m. Hence, we obtain the coordinate representation of the vector field v via the map $y\colon \Omega := \varphi(U) \to \mathbb{R}^k$. Then $\gamma\colon (a,b) \to U$ is an integral curve of v if and only if for $\Gamma := \varphi \circ \gamma$ we have

$$\dot{\Gamma}(t) = y\big(\Gamma(t)\big) \qquad \text{for } \ t \in (a,b),$$

cf. (2.76). By Theorem 2.78, there exists a neighborhood V_m such that for each $\hat{m} \in V_m$ the integral curve $\gamma_{\hat{m}}$ through $\hat{m}$ is defined on $(-\varepsilon_m, \varepsilon_m)$. By the compactness of M, we can find a finite collection of points $\{m_1, \dots, m_\ell\}$ such that the corresponding collection of neighbourhoods $\big\{V_j := V_{m_j} \mid 1 \le j \le \ell\big\}$ covers all of M. Set

$$\varepsilon := \frac{\min\{\varepsilon_{m_j} \mid 1 \le j \le \ell\}}{2} > 0.$$

Let $\gamma\colon (a,b) \to M$ be a maximal integral curve of v. Assuming $b < \infty$, the point $m_0 := \gamma(b-\varepsilon)$ lies in some V_j. By the construction of ε, there is a unique integral curve γ_{m_0}, which is well-defined on $(-2\varepsilon, 2\varepsilon)$ and satisfies $\gamma_{m_0}(0) = m_0$. Set

$$\hat{\gamma}\colon (a,\, b+\varepsilon) \to M, \qquad \hat{\gamma}(t) = \begin{cases} \gamma(t) & \text{for } t \in (a,\, b-\varepsilon), \\ \gamma_{m_0}\big(t - b + \varepsilon\big) & \text{for } t \in \big[b-\varepsilon,\, b+\varepsilon\big). \end{cases}$$

Notice that $\hat{\gamma}$ is continuous since $\gamma_{m_0}(b-\varepsilon) = m_0 = \gamma(b-\varepsilon)$. In fact, by construction $\hat{\gamma}$ is an integral curve of v on $(a, b-\varepsilon) \cup (b-\varepsilon, b+\varepsilon)$. It follows that $\hat{\gamma}$ is a C^1-integral curve of v and therefore smooth by Exercise 2.77. Thus, $\hat{\gamma}$ is an integral curve of v defined on a larger interval. This contradicts the maximality of γ. ☐

2.10 Flows and 1-parameter groups of diffeomorphisms

In this section I assume that M is a compact manifold.

For a vector field v define *the flow* of v to be the map

$$\Phi\colon \mathbb{R} \times M \to M, \qquad \Phi(t,m) = \gamma_m(t).$$

Of course, this is just the map Φ of Theorem 2.78 extended to the whole real line. Sometimes, (2.79) is referred to as *the local flow* of v.

Beside the flow, for each fixed $t \in \mathbb{R}$ it is also convenient to consider

$$\Phi_t\colon M \to M, \qquad \Phi_t(m) = \Phi(t,m) = \gamma_m(t).$$

Proposition 2.82. *The following holds:*

(i) *Each Φ_t is a diffeomorphism. Moreover, $\Phi_t^{-1} = \Phi_{-t}$;*

(ii) *For any $t, s \in \mathbb{R}$ we have $\Phi_t \circ \Phi_s = \Phi_{t+s} = \Phi_s \circ \Phi_t$;*

(iii) $\Phi_0 = id_M$;

Proof. For $m \in M$ and $t \in \mathbb{R}$ denote $\Phi_t(m) = \hat{m}$. This means that $\gamma_m(t) = \hat{m}$, where γ_m is an integral curve of v such that $\gamma_m(0) = m$.

Consider the curve β defined by $\beta(s) = \gamma_m(s+t)$. Then β is an integral curve of v and $\beta(0) = \gamma_m(t) = \hat{m}$, that is $\beta = \gamma_{\hat{m}}$. Hence,

$$\Phi_s(\hat{m}) = \gamma_{\hat{m}}(s) = \beta(s) = \gamma_m(s+t) = \Phi_{s+t}(m) \qquad \Longleftrightarrow \qquad \Phi_s \circ \Phi_t = \Phi_{s+t}.$$

Since *(iii)* holds by the very definition of Φ_t, by *(ii)* we obtain

$$\Phi_{-t} \circ \Phi_t = id_M = \Phi_t \circ \Phi_{-t}.$$

In particular, each Φ_t is a diffeomorphism and $\Phi_t^{-1} = \Phi_{-t}$ □

Definition 2.83. A 1-parameter group of diffeomorphisms is any smooth map $\Phi \colon \mathbb{R} \times M \to M$ such that Properties *(i)*–*(iii)* of Proposition 2.82 hold.

To explain the above definition, notice that the set

$$\mathrm{Diff}(M) := \big\{ f \colon M \to M \mid f \text{ is a diffeomorphism} \big\}$$

is a group with respect to the composition operation. $\mathrm{Diff}(M)$ is called *the diffeomorphism group* of M. With this understood, a 1-parameter group of diffeomorphisms is simply a homomorphism of groups

$$\mathbb{R} \to \mathrm{Diff}(M), \qquad t \mapsto \Phi_t$$

such that $\Phi_t(m) = \Phi(t, m)$ depends smoothly on (t, m).

Thus, Proposition 2.82 states that each vector field on a compact manifold generates a 1-parameter group of diffeomorphisms. Conversely, it turns out that any 1-parameter group of diffeomorphisms generates a vector field in the following sense.

Proposition 2.84. *For any 1-parameter group of diffeomorphisms Φ there exists a vector field v, whose 1-parameter group of diffeomorphisms coincides with Φ.*

Proof. For any $m \in M$ denote

$$\gamma_m \colon \mathbb{R} \to M, \qquad \gamma_m(t) := \Phi(t, m) \qquad \text{and} \qquad v(m) := \dot{\gamma}_m(0).$$

The reader should check that v is a smooth vector field.

Furthermore, denote $\gamma_m(t) = \hat{m}$ and observe that

$$\gamma_{\hat{m}}(s) = \Phi_s(\hat{m}) = \Phi_s\big(\Phi_t(m)\big) = \Phi_{t+s}(m) = \gamma_m(t+s). \tag{2.85}$$

Hence,

$$v\big(\gamma_m(t)\big) = v\big(\hat{m}\big) = \dot{\gamma}_{\hat{m}}(0) = [\gamma_{\hat{m}}(s)]_{s=0} = [\gamma_m(t+s)]_{s=0} = \dot{\gamma}_m(t),$$

where it is straight-forward to obtain all above equalities from the corresponding definitions. Thus, γ_m is an integral curve of v. Therefore, the 1-parameter group of diffeomorphisms generated by v is

$$(t, m) \mapsto \gamma_m(t) = \Phi(t, m),$$

In other words, the 1-parameter group of diffeomorphisms generated by v coincides with Φ. □

To sum up, for compact manifolds there is a natural bijective correspondence between vector fields and 1-parameter groups of diffeomorphisms.

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