List of Problems in Global Analysis

1. Let M be a closed oriented Riemannian manifold. Show that any solution $\omega \in \Omega^k(M)$ of the equation $\Delta\omega = d\eta$, where $\eta \in \Omega^{k-1}(M)$, is closed.

2. Prove that any cohomology class in $H^1_{dR}\big(\mathbb{R}^2 \setminus \{0\}\big)$ is represented by a harmonic 1-form.

3. Prove that on any closed connected oriented Riemannian n-manifold, any harmonic n-form is proportional to the volume form.

4. Let D be the disc in $\mathbb{R}^2$ of unit radius centered at the origin. Find the dimension of the space
$$\big\{\omega \in \Omega^1(D) \mid d\omega = 0 = d^*\omega,\ \omega(\partial_n) = 0\big\},$$
where ∂_n is the unit normal field along ∂D. Also, show that the space
$$\big\{\omega \in \Omega^1(D) \mid \Delta\omega = 0,\ \omega(\partial_n) = 0\big\},$$
is infinite dimensional.

5. Let M^2 be an oriented surface equipped with a Riemannian metric g. For an everywhere positive function u, denote $g_u := u^2 g$. Show that if $\omega \in \Omega^k(M)$, $k = 0, 1, 2$, we have
$$\Delta_g\omega = 0 \qquad \Longleftrightarrow \qquad \Delta_{g_u}\omega = 0.$$

6. Show that a k-form ω is harmonic if and only if $*\omega$ is harmonic.

7. Let M be a closed oriented Riemannian four-manifold. Since for the $*$-operator acting on $\Lambda^2 T^*M$ we have $*^2 = id$, the bundle of 2-forms splits into the ± 1-eigenspaces: $\Lambda^2 T^*M = \Lambda^2_+ T^*M \oplus \Lambda^2_- T^*M$.

   (a) Show that $\dim \Lambda^2_+ T^*_m M = \dim \Lambda^2_- T^*_m M$ for each $m \in M$;

   (b) Show that $H^2_{dR}(M)$ splits as $H^2_+(M) \oplus H^2_-(M)$, where $H^2_\pm(M)$ is the maximal positive/negative subspace of the symmetric bilinear form
   $$(\omega, \eta) \mapsto \int_M \omega \wedge \eta;$$

   (c) Show that the sequence
   $$0 \longrightarrow \Omega^0(M) \xrightarrow{\ d\ } \Omega^1(M) \xrightarrow{\ d^\pm\ } \Omega^2_\pm(M) \to 0 \tag{1}$$
   is a complex, whose homology groups are isomorphic to
   $$H^0(M), \qquad H^1(M), \qquad \text{and} \qquad H^2_\pm(M).$$

8. Denote by $\mathcal{H}^k(M)$ the space of all harmonic k-forms on a closed oriented Riemannian manifold M. Assuming that $\mathcal{H}^k(M)$ is finite-dimensional, show that for any we have the decomposition
$$\Omega^k(M) = \operatorname{Im} d \oplus \mathcal{H}^k \oplus \operatorname{Im} d^*,$$
which is in fact L^2-orthogonal.

9. Let Σ be a Riemann surface.

   (i) Show that for any holomorphic $(1,0)$ form ζ, the real 1-forms $\operatorname{Re}\zeta$ and $\operatorname{Im}\zeta$ are harmonic.

   (ii) Show that for any real harmonic 1-form ω there exists a holomorphic $(1,0)$ form ζ such that $\operatorname{Re}\zeta = \omega$.

10. Let M and N be two closed orientable manifolds of the same dimension n. Assume, moreover, that N is connected. Pick $\omega \in \Omega^n(N)$ such that $\int_N \omega = 1$ and define $\deg f = \int_M f^*\omega$.

    (i) Show that $\deg f$ is well-defined;

    (ii) Show that $\deg f$ is in fact an integer;

    (iii) Show that f is surjective whenever $\deg f \neq 0$;

    (iv) Let $f\colon \Sigma_1 \to \Sigma_2$ be a holomorphic map between compact Riemann surfaces and let $F(z)$ be a coordinate representation of f with respect to a local holomorphic coordinate z centered at some $p \in \Sigma_1$ (and a local holomorphic coordinate on Σ_2). A non-negative integer $m = m(p)$ is said to be a multiplicity of f at p if F can be represented in the form $F(z) = F(0) + z^m F_1(z)$, where $F_1(0) \neq 0$. Show that for *any* $q \in \Sigma_2$ we have
    $$\deg f = \sum_{p \in f^{-1}(q)} m(q).$$

    (v) Let $f\colon \Sigma_1 \to \Sigma_2$ be a holomorphic map between compact Riemann surfaces. Show that $\deg f \geq 0$. Moreover, $\deg f = 0$ if and only if f is constant and $\deg f = 1$ if and only if f is a biholomorphism.

    (vi) Prove the following: If a compact Riemann surface Σ admits a meromorphic function with a unique simple pole, then Σ is biholomorphic to $\mathbb{CP}^1$.

11. Let Σ be a Riemann surface diffeomorphic (homeomorphic) to the torus. Prove that for any two distinct points $p_1, p_2 \in \Sigma$ there exists a meromorphic function f on Σ such that both p_1 and p_2 are simple poles of f and f is holomorphic on $\Sigma \setminus \{p_1, p_2\}$. Also, show that the following limiting case holds: for any $p \in \Sigma$ there exists a meromorphic function g with a unique pole at p of order 2.

12. Prove that the wedge-product of harmonic forms does not need to be harmonic (*Hint:* Take a compact Riemann surface Σ of genus ≥ 2. Pick a non-trivial holomorphic $(1,0)$ form ζ. Show that $\operatorname{Re}\zeta \wedge \operatorname{Im}\zeta \neq 0$ must vanish somewhere and therefore cannot be harmonic.)

13. Prove that the tangent bundle of the 2-sphere is non-trivial.

14. Show that there is a non-trivial bundle $E \to M$ such that $E \oplus \underline{\mathbb{R}}^k$ is trivial, where $\underline{\mathbb{R}}^k$ denotes the trivial vector bundle of rank k.

15. Let $E \to I = [0,1]$ be a vector bundle.

    (a) Pick a connection ∇ on E. Show that for any $\mathrm{v} \in E_0$ there exists a unique section s_{v} such that $\nabla s_{\mathrm{v}} = 0$ and $s_{\mathrm{v}}(0) = \mathrm{v}$.

    (b) Show that any bundle $E \to I$ is trivial.

(c) Show that for any vector bundle $E \to M \times I$ we have $E \cong \pi_1^* E|_{M\times\{0\}} = E|_{M\times\{0\}} \times I$, where $\pi_1 \colon M \times I \to M$ is the natural projection.

(d) Let $f_0, f_1 \colon M \to N$ be two smoothly homotopic maps. Show that $f_0^* E \cong f_1^* E$ for any vector bundle $E \to N$.

(e) Show that any vector bundle over a contractible base is trivial.

16. Denote

$$L = \big\{([z], w) \in \mathbb{CP}^1 \times \mathbb{C}^2 \mid w = 0 \text{ or } [w] = [z]\big\}.$$

Define the projection map $\pi \colon L \to \mathbb{CP}^1$ by $\big([z], w\big) \mapsto [z]$. Show that L is a complex vector bundle of rank 1 over $\mathbb{CP}^1 \cong S^2$. This is called the tautological line bundle of $\mathbb{CP}^1$.

17. Let L be a complex line bundle bundle, that is a complex vector bundle of rank 1, over S^2 such that L admits a trivialization σ_N over $S^2 \setminus \{N\}$ and a trivialization σ_S over $S^2 \setminus \{S\}$, where $N = -S$ is the northern pole[1]. This yields a map $g \colon S^2 \setminus \{S, N\} \to \mathbb{C}^*$ defined by

$$\sigma_S(m) = g(m)\sigma_N(m).$$

The degree of the map $g/|g| \colon S^1 \to S^1$, where the source $S^1 \subset S^2 \setminus \{S, N\}$ is thought of as the equator, is called the degree of L. Show that the following holds:

(i) The degree of a complex line bundle is well-defined and depends on the isomorphism class of L only.

(ii) The degree of the tautological bundle equals -1.

(iii) The degree of T^*S^2 equals 2. Here T^*S^2 is viewed as a complex line bundle as follows: The Hodge operator on T^*S^2 satisfies $*^2 = -id$. Hence, elements of T^*S^2 can be multiplied by complex numbers: $(a + bi) \cdot \omega := a\omega + b * \omega$.

(iv) $\deg(L_1 \otimes L_2) = \deg L_1 + \deg L_2$.

(v) $\deg L^* = -\deg L$, where $L^* = \operatorname{Hom}(L, \underline{\mathbb{C}})$ is the dual line bundle.

(vi) For any integer n there exists a complex line bundle L_n such that $\deg L_n = n$.

(vii) Two line bundles are isomorphic if and only if their degrees are equal.

(viii) Prove that the tangent bundle of S^2 is non-trivial.

18. (a) Let $E \to M$ be a vector bundle and $F \to M$ be a subbundle of E. Show that there is a subbundle $G \subset E$ such that $E = F \oplus G$. In other words, any short exact sequence of vector bundles $0 \to F \to E \to G \to 0$ splits.

(b) Prove that for any vector bundle E over a compact[2] base M there exists a vector bundle F such that $E \oplus F$ is trivial.

(c) Denote by $\operatorname{Gr}_k(\mathbb{R}^n)$ the Grassmannian of all k-dimensional subspaces in $\mathbb{R}^n$. Let

$$E_{k,n} := \big\{(x, V) \in \mathbb{R}^n \times \operatorname{Gr}_k(\mathbb{R}^n) \mid x \in V\big\}$$

be the tautological bundle over $\operatorname{Gr}_k(\mathbb{R}^n)$, cf. Problem 16. Show that for any rank k vector bundle E over a compact manifold M there exists a (smooth) map $f \colon M \to \operatorname{Gr}_k(\mathbb{R}^n)$ such that $E \cong f^* E_{k,n}$.

[1] In fact any vector bundle has this property by Problem 15 (e).
[2] This is true for any manifold, however, a solution is slightly simpler if one assumes the base to be compact.

19. Show that any function $f \in H^1(0,1)$ is continuous without using the Sobolev embedding theorem.

20. Show that the function

    (i) $f(x) = |x|$ belongs to $H^1(-1,1)$;

    (ii) $f(x) = |x|^{1/2}$ does not belong to $H^1(-1,1)$.

21. For which values of $a \in \mathbb{R}$ does the function $f(x) = |x|^a$ belong to $H^k(\mathbb{R}^n)$?

22. Show that there exists a function $f \in H^1(\mathbb{R}^2)$, which is not continuous.

23. (a) Prove the following simple version of the Sobolev embedding theorem: Show that $H^1(0,1)$ embeds into the Hölder space $C^{0,1/2}(0,1)$.

    (b) Prove that the embedding $H^1(0,1) \to C^0(0,1)$ is compact.

24. Let $q\colon H \times H \to \mathbb{R}$ be a symmetric bilinear form on a Hilbert space H with the following property: for any $u \in H$ there is some constant $C(u) > 0$ such that $|q(u,v)| \le C(u)\|v\|$ for all $v \in H$. Show that there exists a symmetric bounded linear operator $Q\colon H \to H$ such that $\langle Qu, v\rangle = q(u,v)$.

25. (a) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain (with a smooth boundary). Prove that the Poincaré inequality
$$\|u\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)}$$
holds for any $u \in C_0^1(\Omega) := \{u \in C^1(\mathbb{R}^n) \mid \operatorname{supp} u \subset \Omega\}$. Here the constant C may depend[3] on Ω, but not on u.
Remark: One can show that for any $u \in H^1(\Omega)$ the trace $u|_{\partial\Omega}$ is well defined as an L^2-function, for example. With this at hand, one can define $H_0^1(\Omega)$ as a subspace of H^1-functions with vanishing trace on the boundary.

    (b) Define $H_0^1(\Omega)$ as a completion of $C_0^1(\Omega)$ with respect to the H^1-norm: $H_0^1(\Omega) := \overline{C_0^1(\Omega)}_{\|\cdot\|_{H^1}}$. Show that
$$a(u,v) := \int_\Omega \langle \nabla u, \nabla v\rangle$$
is a scalar product on $H_0^1(\Omega)$ equivalent to the standard one, i.e., there exist positive constants c and C such that
$$c\|u\|_{H^1} \le a(u,u)^{1/2} \le C\|u\|_{H^1}$$
holds for all $u \in H_0^1(\Omega)$.

    (c) A function $u \in H_0^1(\Omega)$ is called *a weak solution* of the Poisson equation
$$\Delta u = f, \qquad u|_{\partial\Omega} = 0$$
if $a(u,\varphi) = \langle f, \varphi\rangle_{L^2}$ holds for any $\varphi \in C_0^\infty(\Omega)$, where $f \in C^0(\bar\Omega)$. Show that any strong solution $u \in C^2(\Omega) \cap C^0(\bar\Omega)$ of the Poisson equation is its weak solution.

    (d) Show that the Poisson equation has a weak solution $u \in H_0^1(\Omega)$.

[3] One can show that C may be chosen to be independent of Ω, but the proof of this is somewhat more elaborate.

(e) Show that the energy functional $E_f\colon H^1_0(\Omega)\to\mathbb{R}$, $E_f(u)=\|\nabla u\|^2_{L^2}-\langle f,u\rangle_{L^2}$ is bounded from below and E_f attains its infimum. Moreover, if u is a point of minimum of E_f and $u\in C^2(\Omega)\cap C^0(\bar\Omega)$, then u is a strong solution of the Poisson equation.

26. Let $E\to M$ be a vector bundle. Compute the symbol of an arbitrary connection ∇ on E.

27. Compute the symbol of $d^*\colon \Omega^{k+1}(M)\to\Omega^k(M)$.

28. For a complex manifold M, compute the symbol of the Dolbeault operator $\bar\partial\colon \Omega^{p,q}(M)\to\Omega^{p,q+1}(M)$.

29. Show that the operator

$$L\colon C^\infty(\mathbb{R}^3;\mathbb{H})\to C^\infty(\mathbb{R}^3;\mathbb{H}),\qquad Lu=i\,\partial_x u+j\,\partial_y u+k\,\partial_z u$$

is elliptic, where $\mathbb{H}$ denotes the algebra of quaternions.

30. Is the bi-Laplacian $u\mapsto\Delta(\Delta u)$, $u\in C^\infty(\mathbb{R}^n)$, an elliptic operator? Is $d+d^*\colon\Omega^k(M)\to\Omega^{k+1}(M)\oplus\Omega^{k-1}(M)$ elliptic? Is $d+d^*\colon\Omega^{\mathrm{even}}(M)\to\Omega^{\mathrm{odd}}(M)$ elliptic, where $\Omega^{\mathrm{even}}(M):=\Omega^0\oplus\Omega^2\oplus\dots$?

31. Show that any pseudo-differential operator acting on $C^\infty_0(\mathbb{R}^n)$, say, is an integral operator, that is of the form

$$u\mapsto\int_{\mathbb{R}^n}K(x,y)u(y)\,dy.$$

Compute K for the inverse of the standard Laplacian on $\mathbb{R}^n$.

32. Let

$$\Gamma(E_0)\xrightarrow{L_0}\Gamma(E_1)\xrightarrow{L_1}\Gamma(E_2)\tag{2}$$

be a complex, where both L_0 and L_1 are differential operators. Show that (2) is an elliptic complex if and only if the operator $L_1+L_0^*\colon\Gamma(E_1)\to\Gamma(E_2)\oplus\Gamma(E_0)$ is elliptic.

33. Show that Atiyah's complex (1) is elliptic.

34. Prove that a bounded linear operator $T\colon H_1\to H_2$, where H_1 and H_2 are Hilbert spaces, is Fredholm if and only if there exist bounded linear maps $S_1,S_2\colon H_2\to H_1$ such that

$$S_1\circ T=\mathrm{id}_{H_1}+R_1\qquad\text{and}\qquad T\circ S_2=\mathrm{id}_{H_2}+R_2,$$

where both R_1 and R_2 are compact.